§10.2

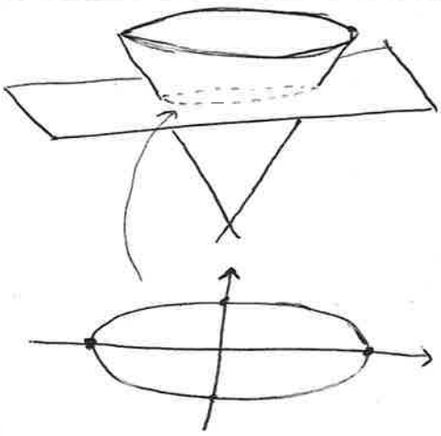
MAIN CONCEPTS

1. Ellipses ($e<1$)

2. Hyperbolae ($e>1$).

---

1 We continue our discussion of conics with the case $e<1$, which gives the elongated circle shown, called an ellipse.



We can do a similar derivation as for the parabola in §10.1, but we'll just skip to the conclusion summary.

ELLIPSE STANDARD FORM: The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with $a\geq b>0$ has a focus at $(c,0)$ and eccentricity e satisfying $b=a\sqrt{1-e^2}$ and $c=\sqrt{a^2-b^2}$.

Notice from the graph and the standard form that the ellipse is doubly symmetric. Flipping over either axis gives the same shape, so $(-c,0)$ is another focus. For ellipses, we usually specify both foci (plural of focus) rather than the directrix.

Second, note that the ellipse has 4 key points on the axes, which we'll call the vertices. When $x=0$ we have $\frac{y^2}{b^2}=1 \longrightarrow y^2=b^2 \longrightarrow y=\pm b$ and when $y=0$ we have $\frac{x^2}{a^2}=1 \longrightarrow x^2=a^2 \longrightarrow x=\pm a$. So the vertices are at $(\pm a,0)$ and $(0,\pm b)$ for standard form.

EXAMPLES: Locate the foci and vertices of each ellipse.

① $\frac{x^2}{25}+\frac{y^2}{9}=1$

This is in standard form with $a^2=25 \longrightarrow a=5$

$b^2=9 \longrightarrow b=3$

$c=\sqrt{a^2-b^2}=\sqrt{25-9}=\sqrt{16}=4.$

So it has vertices $(\pm 5,0)$ and $(0,\pm 3)$, and foci at $(\pm 4,0)$.

② $x^2+5y^2=5$

This isn't quite in standard form, so we first divide by 5: $\frac{x^2}{5}+y^2=1$ or $\frac{x^2}{5}+\frac{y^2}{1}=1$.

So $a^2=5 \longrightarrow a=\sqrt{5}$ vertices: $(\pm\sqrt{5},0)$

$b^2=1 \longrightarrow b=1$ vertices: $(0,\pm 1)$

$c=\sqrt{a^2-b^2}=\sqrt{5-1}=\sqrt{4}=2$ foci: $(\pm 2,0)$.

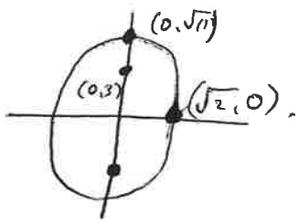
③ $\frac{x^2}{2} + \frac{y^2}{11} = 1$

This looks like standard form, but there's an issue: $a^2 = 2$ and $b^2 = 11$ so $b^2 > a^2$ and we're supposed to have $a^2 \geq b^2$. In fact, this would give $c = \sqrt{2-11} = \sqrt{-9}$ which doesn't exist. Like in §10.1, we address this by swapping x and y.

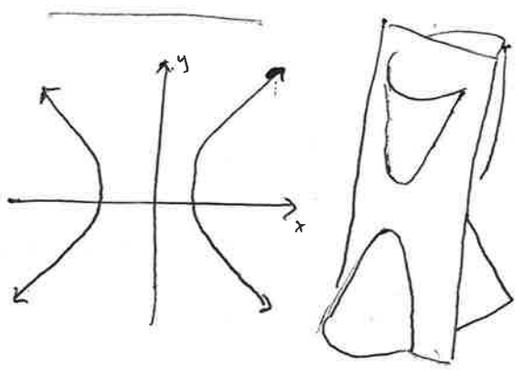
Modify $(x \leftrightarrow y)$: $\frac{y^2}{2} + \frac{x^2}{11} = 1$
- $a^2 = 11 \rightarrow a = \sqrt{11} \rightarrow$ vertices at $(\pm\sqrt{11}, 0)$
- $b^2 = 2 \rightarrow b = \sqrt{2} \rightarrow$ vertices at $(0, \pm\sqrt{2})$
- $c = \sqrt{11-2} = \sqrt{9} = 3 \rightarrow$ foci at $(\pm 3, 0)$

Unmodify $(x \leftrightarrow y)$: Vertices at $(0, \pm\sqrt{11})$
Vertices at $(\pm\sqrt{2}, 0)$
Foci at $(0, \pm 3)$

Visually, we have:



2 Finally, we handle $e > 1$, which, unlike the previous cases, has two "branches." This shape is called a hyperbola. Solving $|PF| = e|PD|$ gives:



HYPERBOLA STANDARD FORM: The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has a focus at $(c, 0)$ and eccentricity e satisfying $b = a\sqrt{e^2-1}$ and $c = \sqrt{a^2+b^2}$, with $a, b > 0$

(Note the slight differences with the ellipse standard form, largely because of $\pm$ changes.)

Again, symmetry gives a second focus at $(-c, 0)$. When $y = 0$, we get

$\frac{x^2}{a^2} = 1 \rightarrow x^2 = a^2 \rightarrow x = \pm a$ so there are vertices at $(\pm a, 0)$. From the graph above, there are no more vertices. Indeed, when $x = 0$, $-\frac{y^2}{b^2} = 1 \rightarrow y^2 = -b^2$ which is impossible.

For sketching/identification, we'd like another identifying feature (two vertices isn't enough). For this, we ask for limiting behavior. We solve for y:

$$\frac{x^2}{a^2} - 1 = \frac{y^2}{b^2} \rightarrow y^2 = \frac{b^2}{a^2}x^2 - b^2 \rightarrow y = \pm\sqrt{\frac{b^2}{a^2}x^2 - b^2}.$$

When x is large, the constant (b^2) is largely irrelevant. So, for x large,

$$y \approx \pm\sqrt{\frac{b^2}{a^2}x^2 - 0} = \pm\sqrt{\frac{b^2}{a^2}x^2} = \pm\frac{b}{a}x.$$

HYPERBOLA ASYMPTOTES: The graph of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has asymptotes along $y = \pm\frac{b}{a}x$.

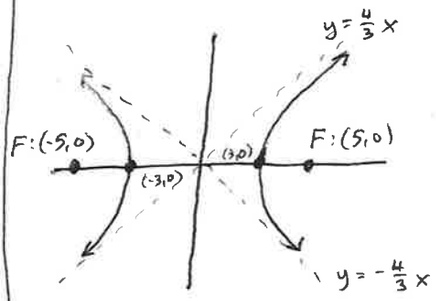
EXAMPLES: Identify the vertices, foci, and asymptotes of each hyperbola and use them to sketch it.

① $\frac{x^2}{9} - \frac{y^2}{16} = 1$

This is in standard form: $a^2 = 9 \longrightarrow a = 3 \longrightarrow$ vertices at $(\pm 3, 0)$

$b^2 = 16 \longrightarrow b = 4 \longrightarrow$ asymptotes along $y = \pm \frac{4}{3}x$

$c = \sqrt{a^2+b^2} = \sqrt{9+16} = \sqrt{25} = 5 \longrightarrow$ foci at $(\pm 5, 0)$.



② $x^2 = 4 + 2y^2$

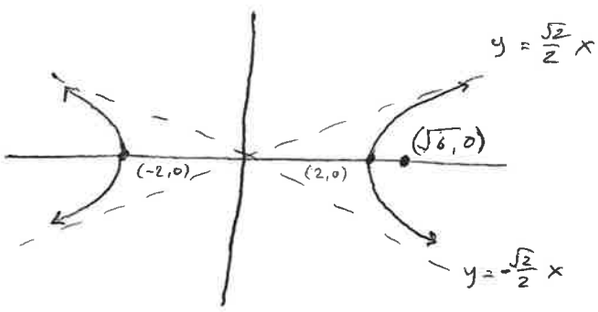
First, put it in standard form:

$x^2 - 2y^2 = 4 \longrightarrow \frac{x^2}{4} - \frac{y^2}{2} = 1$

So: $a^2 = 4 \longrightarrow a = 2 \longrightarrow$ vertices at $(\pm 2, 0)$

$b^2 = 2 \longrightarrow b = \sqrt{2} \longrightarrow$ asymptotes along $y = \pm \frac{\sqrt{2}}{2}x$

$c = \sqrt{4+2} = \sqrt{6} \longrightarrow$ foci at $(\pm\sqrt{6}, 0)$



③ $x^2 = \frac{y^2}{3} - 1$

We try to put it in standard form: $x^2 - \frac{y^2}{3} = -1$

$\longrightarrow \frac{y^2}{3} - \frac{x^2}{1} = 1$. Again we see that this is "reverse" standard form by flipping x & y. So we get vertices $(0, \pm\sqrt{3})$, asymptotes $y = \pm\sqrt{3}\cdot x$, and foci $(0, \pm 2)$.

