§10.3

MAIN CONCEPTS

1. General form of conics
2. Translations
3. Degenerate conics
4. Rotations (briefly)

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1. From the previous sections, we've seen that conic sections involve powers of x and y up to degree 2. In general, then, the overall possibility for a conic section is:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Standard form for a parabola is: $A = B = E = F = 0$, $C = 1$, $D = -4p$, so $y^2 - 4px = 0 \to y^2 = 4px$.

—"— an ellipse is: $A = \frac{1}{a^2}$, $C = \frac{1}{b^2}$, $F = -1$, $B = D = E = 0$, so $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \to \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

—"— a hyperbola is: $A = \frac{1}{a^2}$, $C = -\frac{1}{b^2}$, $F = -1$, $B = D = E = 0$, so $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0 \to \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

2. All of the above had $E = 0$. Let's consider $D, E \neq 0$. Then, by completing the square, we can recognize non-standard conics.

EXAMPLES: Identify each conic section, and locate a focus.

① $4x^2 + y^2 + 40x - 2y + 97 = 0$

Group the x & y terms and complete the square: $4(x^2 + 10x) + (y^2 - 2y) = -97$

$\to 4(x^2 + 10x + 25) + (y^2 - 2y + 1) = -97 + 4 \cdot 25 + 1 \longrightarrow 4(x+5)^2 + (y-1)^2 = 4$

$\to (x+5)^2 + \frac{(y-1)^2}{4} = 1$. So this is an ellipse, shifted to be centered at $(-5, 1)$.

The ellipse is also rotated, so $c = \sqrt{4-1} = \sqrt{3}$ gives foci in the y-direction, i.e. the foci are at $(-5, 1 \pm \sqrt{3})$.

② $y^2 - 4x - 12y + 28 = 0$

Ditto: $(y^2 - 12y) - 4x = -28 \longrightarrow y^2 - 12y + 36 = 4x - 28 + 36 \to (y-6)^2 = 4x + 8$

$\to (y-6)^2 = 4(x+2)$. This is a parabola with vertex $(-2, 6)$ and $p = 1$ since the coefficient of x is $4p = 4 \to p = 1$. So the focus is $(1, 0)$ translated from the vertex, i.e. $(-2, 6+1) = (-2, 7)$.

③ $16x^2 - 9y^2 + 192x + 90y - 495 = 0$

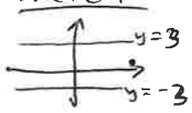
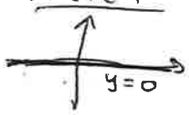
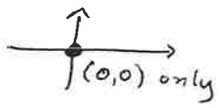
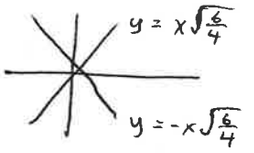
Ditto: $16(x^2 + 12x) - 9(y^2 - 10y) = 495 \longrightarrow 16(x^2 + 12x + 36) - 9(y^2 - 10y + 25) = 495 + 16 \cdot 36 - 9 \cdot 25$

$\to 16(x+6)^2 - 9(y-5)^2 = 846 \longrightarrow \frac{(x+6)^2}{423/8} - \frac{(y-5)^2}{94} = 1$

This is a hyperbola in the x-direction centered at $(-6,5)$. Then $c = \sqrt{(\frac{423}{8})^2 + 94^2} \approx 12.12$.

So the foci are at $(-6 \pm 12.12, 5)$.

3 Even after completing the square, we might be left with something irreconcilable with standard conics. E.g. the equation $y^2 = 4$ gives $y = \pm 2$ which isn't a conic. Examples like these are called <u>degenerate conics</u>. After completing the square, the degenerate conics we can have are of the form:

Equation	Sketch	Equation	Sketch
• $y^2 = 9$		• $y^2 = 0$	
• $y^2 = -1$ (empty; no solution)		• $\frac{x^2}{2} + y^2 = 0$	
• $\frac{x^2}{2} + \frac{y^2}{7} = -1$ (empty)		• $\frac{x^2}{4} - \frac{y^2}{6} = 0$	

EXAMPLE: Identify the conic $4x^2 - y^2 - 8x - 6y - 5 = 0$

Do the same: $4(x^2-2x) - (y^2+6y) = 5 \longrightarrow 4(x^2-2x+1) - (y^2+6y+9) = 5 + 4\cdot 1 - 9$

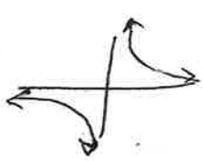
$\longrightarrow 4(x-1)^2 - (y+3)^2 = 0 \longrightarrow (x-1)^2 - \frac{(y+3)^2}{4} = 0$.

This looks like a hyperbola, but we can't change the zero to a one, so it is degenerate (intersecting lines case).

4 Finally, throughout this discussion, we've left $B = 0$. Rather than explain how to handle $B \neq 0$ in general, let's look at a specific example:

EXAMPLE: Identify the conic section $xy = 1$.

To help us, we can sketch the graph. This is $y = \frac{1}{x}$ which we've seen:



It has two branches, so it's probably a hyperbola, but <u>rotated</u>. To unrotate, let's switch to new axes: the $u = \frac{x+y}{2}$ axis and $v = \frac{x-y}{2}$ axis. Then

$x = u + v$ and $y = u - v$, so $xy = 1 \longrightarrow (u+v)(u-v) = 1$

$\longrightarrow u^2 - v^2 = 1$ which is indeed a hyperbola.

In general, we've learned that in the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$,

i) A & C control which conic section it is,

ii) D & E control translation (where it is centered),

iii) F controls scaling and degeneracy, and

iv) B controls rotation.