§10.4

Main concepts

1. Definition of parametrization
2. Recognizing a curve from its parametrization
3. Chain Rule for differentiation
4. Substitution for integration

1

Definition: A plane curve is parametrized by

$$x = f(t), \quad y = g(t). \qquad t \text{ in } I. \quad (t \in I)$$

We often think of "t" as "time", so $x(t)$ is the x-coordinate at time t and $y(t)$ is the y-coordinate at time t.

"I" is an interval, specifying the start/end times. Often, we'll use $I = [0,1]$, so $t=0$ is the start and $t=1$ is the end or $I = \mathbb{R} = (-\infty, \infty)$, so there is no start or end.

Examples: ① $x(t) = t^2 + 2t + 2$ $\quad y(t) = t+1$ $\quad t \in \mathbb{R}$

② $x(t) = \sin(2\pi t)$ $\quad y(t) = t^2$ $\quad t \in [0,1]$

③ $x(t) = 2\cos(t)$ $\quad y(t) = 2\sin(t)$ $\quad t \in [0, 2\pi]$

④ $x(t) = \log(t)$ $\quad y(t) = t^2$ $\quad t \in (0, \infty)$

2

- Graphing is an invaluable tool. Some recommendations for graphing a plane curve from the parametrization:

① Plot endpoints

② Plot critical/important points

③ Consider the sign

④ Fill in by gaps.

Let's apply this to graphing example ②:

Endpoints: $t=0 \longrightarrow x=\sin(0)=0$, $y=0^2=0$ $\quad (0,0)$

$t=1 \longrightarrow x=\sin(2\pi)=0$, $y=1^2=1$ $\quad (0,1)$

Critical: When is $\sin(2\pi t)=0$? $2\pi t = \ldots, -\pi, 0, \pi, 2\pi, \ldots$

$t = \ldots, -\frac{1}{2}, 0, \boxed{\frac{1}{2}}, 1, \ldots$

$x(\frac{1}{2})=0$, $y(\frac{1}{2})=(\frac{1}{2})^2=\frac{1}{4}$ $\quad (0,\frac{1}{4})$

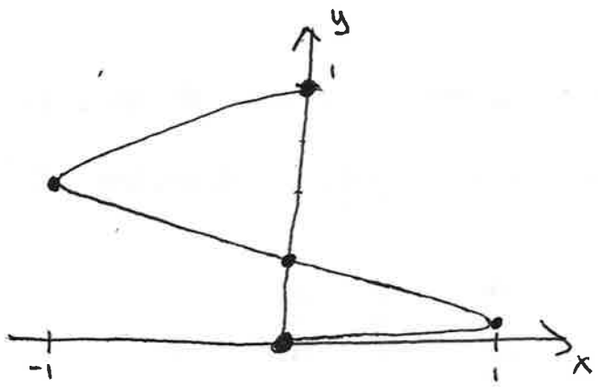
When is $\sin(2\pi t)$ max? $t=\frac{1}{4} \longrightarrow x=\sin(\frac{\pi}{2})=1$

$y=(\frac{1}{4})^2=\frac{1}{16}$ $\quad (1, 1/16)$

When is $\sin(2\pi t)$ min? $t=\frac{3}{4} \longrightarrow x=\sin(\frac{3\pi}{2})=-1$

$y=(\frac{3}{4})^2=9/16$ $\quad (-1, 9/16)$

Graph:



- We can also recognize a parametrization algebraically by eliminating the parameter "t". When we do this we <u>need to</u> separately remember <u>I</u>, the interval.

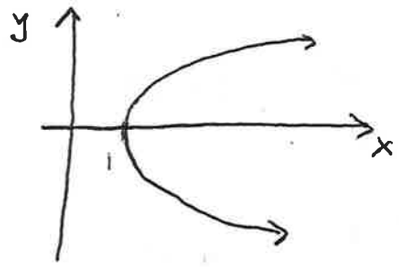
Example ①: $y=t+1$, so $t=y-1$.

Plug this into x: $x=t^2+2t-2=(y-1)^2+2(y-1)+2$

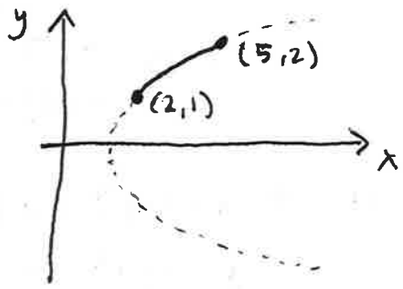
$=(y^2-2y+1)+(2y-2)+2$

$=y^2+1$.

So, this looks like a sideways parabola:



How does $t \in \mathbb{R}$ affect this? Since $y = t+1$, y can be any number, so we get the whole parabola. What if $t \in [0,1]$ instead?



$t=0 \rightarrow x = 0^2+2\cdot 0+2 = 2$, $y = 0+1 = 1$ $\quad (2,1)$

$t=1 \rightarrow x = 1^2+2\cdot 1+2 = 5$, $y = 1+1 = 2$ $\quad (5,2)$.

Now we just get this segment of the parabola.

3

By eliminating t, we can write x and y in terms of each other, so we can compute $\frac{dy}{dx}$ directly. Can we do it <u>without</u> eliminating t?

Yes: by the Chain rule, $\frac{dy}{dt} = \frac{dx}{dt} \cdot \frac{dy}{dx}$, so

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.}$$

Let's try both for example (4):

<u>Eliminate</u>

$x = \log(t) \rightarrow t = e^x$

$y = t^2 \rightarrow y = (e^x)^2 = e^{2x}$

so

$\frac{dy}{dx} = 2e^{2x}$.

<u>Don't eliminate</u>

$x = \log(t) \rightarrow \frac{dx}{dt} = \frac{1}{t}$

$y = t^2 \rightarrow \frac{dy}{dt} = 2t$

so

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{1/t} = 2t^2$

Q: Why did we get different answers?

A: We didn't, really, since $t = e^x$, so

$$2t^2 = 2(e^x)^2 = 2e^{2x}$$ are the same.

This example worked either way.
On the other hand, some are hard to eliminate t. For ex. (3), eliminating t gives:

$$y = 2\sin\left(\cos^{-1}\left(\frac{x}{2}\right)\right)$$

and we could simplify / take a derivative from here. Alternatively,

$$\frac{dy}{dt} = 2\cos(t), \quad \frac{dx}{dt} = -2\sin(t), \text{ so}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos(t)}{-2\sin(t)} = -\cot(t).$$

Note, we can also take higher derivatives by the Chain Rule:

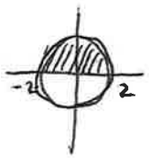
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}.$$

For example (3) again, this gives:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(-\cot(t)\right)}{(-2\sin(t))} = \frac{\csc^2(t)}{-2\sin(t)} = -\frac{1}{2}\csc^3(t).$$

4

Finally, we can also integrate along curves. Notice that ex. (3) is a circle of radius 2. So, the area of the top half of the circle can be calculated with an integral:



$$\int_{-2}^{2} y\,dx$$ (Notice the bounds $x=-2$, $x=2$).

To solve this, substitute. We have $y = 2\sin(t)$ and $x = 2\cos(t)$ so $dx = -2\sin(t)\,dt$. For the bounds, we have

$$x = -2 \longrightarrow -2 = 2\cos(t) \longrightarrow \cos(t) = -1 \longrightarrow t = \pi.$$

$$x = 2 \longrightarrow 2 = 2\cos(t) \longrightarrow \cos(t) = 1 \longrightarrow t = 0$$

So, overall, our integral becomes:

$$\int_{\pi}^{0} (2\sin(t))(-2\sin(t)\,dt) = -4\int_{\pi}^{0} \sin^2(t)\,dt = 4\int_{0}^{\pi} \left(\frac{1-\cos(2t)}{2}\right) dt$$

$$= 2\left[\int_0^{\pi} dt - \int_0^{\pi} \cos(2t)\,dt\right] = 2\left[(\pi - 0) - \tfrac{1}{2}\sin(2t)\Big|_0^{\pi}\right]$$

$$= 2\pi - 0 = 2\pi.$$

On the other hand, we know the area is half of a circle, so should be

$$\tfrac{1}{2}(\pi r^2) = \tfrac{1}{2}\pi(2)^2 = 2\pi.$$

So our calculation matches.

More generally, we can integrate any function of x & y in the same way. Using example ①,

$$\int_1^5 xy\,dy = \int_0^4 (t^2+2t+2)(t+1)\,dt$$

$$= \ldots$$

(finish this example for practice).

Scratch work:

$x = t^2 + 2t + 2$

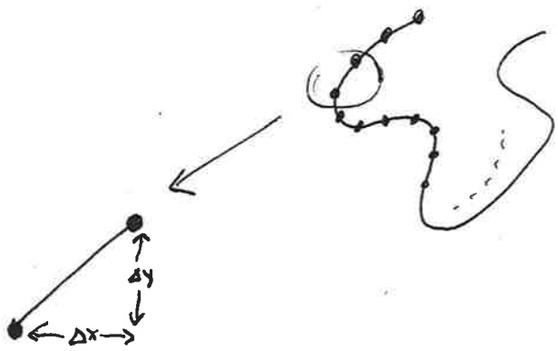
$y = t + 1$

$dy = dt$

$y = 1 \longrightarrow 1 = t+1 \longrightarrow t = 0$

$y = 5 \longrightarrow 5 = t+1 \longrightarrow t = 4$

An important special case of integrating along a curve is computing <u>arc length</u>. To do this, we chop up the curve into "straight" lines of side lengths Δx, Δy and add them up:



So, the total arc length is $\sum \sqrt{\Delta x^2 + \Delta y^2}$. Dividing and multiplying by Δt gives $L = \sum \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\,\Delta t$ and taking the limit makes this an integral:

$$\boxed{L = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.}$$

We can again see this in action with example (3), the circle of radius 2. We have

$$\frac{dx}{dt} = -2\sin(t) \qquad \frac{dy}{dt} = 2\cos(t) \qquad t \in [0, 2\pi], \text{ so}$$

$$L = \int_0^{2\pi} \sqrt{(-2\sin(t))^2 + (2\cos(t))^2}\,dt$$

$$= \int_0^{2\pi} \sqrt{4(\sin^2(t) + \cos^2(t))}\,dt$$

$$= \int_0^{2\pi} \sqrt{4}\,dt$$

$$= \int_0^{2\pi} 2\,dt = 2(2\pi - 0) = 4\pi.$$

And this matches our expectation for the circumference:

$$2\pi r = 2\pi(2) = 4\pi.$$