§10.7

MAIN CONCEPTS

1. Derivatives in polar
2. Integrals in polar

---

1. Strictly speaking, there's no difference differentiating in polar. The problem is one of interpretation: what does $\frac{dr}{d\theta}$ mean? Instead, even in polar, we want the slope of a tangent line, which is $\frac{dy}{dx}$. We can get this from the Chain rule:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$ So we compute each of these. First,

$y = r\sin\theta \longrightarrow \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$, and second,

$x = r\cos\theta \longrightarrow \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$. So, overall,

POLAR TANGENT SLOPE: $$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

EXAMPLES: Compute $\frac{dy}{dx}$ at the specified value of θ.

① $r = 3$ at $\theta = -\pi/4$.

We have $\frac{dr}{d\theta} = 0$, so $\frac{dy}{dx} = \frac{0\sin\theta + 3\cos\theta}{0\cos\theta - 3\sin\theta} = \frac{3\cos\theta}{-3\sin\theta} = -\frac{\cos\theta}{\sin\theta}$. At $\theta = -\pi/4$, we get $\frac{dy}{dx} = -\frac{(\sqrt{2}/2)}{(-\sqrt{2}/2)} = \boxed{1}$.

② $r = 2\cos(\theta)$ at $\theta = \pi/4$.

Here, $\frac{dr}{d\theta} = -2\sin\theta$, so $\frac{dy}{dx} = \frac{-2\sin^2\theta + 2\cos^2\theta}{-2\sin\theta\cos\theta - 2\cos\theta\sin\theta} = \frac{\cos^2\theta - \sin^2\theta}{-2\sin\theta\cos\theta} = \frac{\sin^2\theta - \cos^2\theta}{2\sin\theta\cos\theta}$.

At $\theta = \pi/4$, we have $\frac{dy}{dx} = \frac{(1/\sqrt{2})^2 - (1/\sqrt{2})^2}{2(1/\sqrt{2})(1/\sqrt{2})} = \frac{1/2 - 1/2}{2(1/2)} = \frac{0}{1} = 0$.

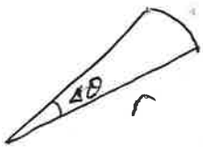
③ $r = \theta$ at $\theta = \pi$

Here, $\frac{dr}{d\theta} = 1$ so $\frac{dy}{dx} = \frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}$. At $\theta = \pi$, we have

$$\frac{dy}{dx} = \frac{(0) + \pi(-1)}{(-1) - \pi(0)} = \frac{-\pi}{-1} = \boxed{\pi}$$

2. Similarly, we can integrate in polar to find areas. In Euclidean, $\int y\,dx$ gives the area under y, but in polar, $\int r\,d\theta$ isn't an area. In fact, it has the wrong units (length, not area). A circular sector of radius r and angle $\Delta\theta$ has

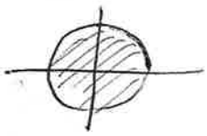
$\Delta A = \frac{1}{2} r^2 \Delta\theta$. Adding these all up and taking the limit as $\Delta\theta \to 0$ gives an integral:



POLAR AREA: The area inside a polar curve is $A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta$.

EXAMPLES: Find the area inside each polar curve.

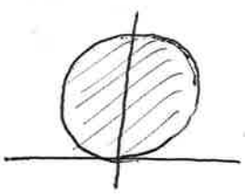
① $r = 2$.



We've seen this: it's a circle, and $0 \le \theta \le 2\pi$. So, the area is $\int_0^{2\pi} \frac{1}{2}(2)^2 d\theta = 2\int_0^{2\pi} d\theta = 2[\theta|_0^{2\pi}] = 2[2\pi - 0] = \boxed{4\pi}$.

This agrees with the formula we know (area $= \pi(\text{radius})^2 = \pi(2)^2 = 4\pi$).

② $r = 2\sin\theta$.

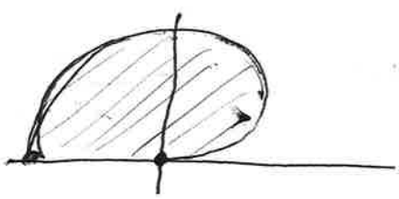


Same idea, but from the graph we see that we only have $0 \le \theta \le \pi$. So, the area is $\int_0^{\pi} \frac{1}{2}(2\sin\theta)^2 d\theta$

$= 2\int_0^{\pi} \sin^2(\theta) d\theta = 2\int_0^{\pi} \frac{1-\cos(2\theta)}{2} d\theta = \theta - \frac{1}{2}\sin(2\theta)\Big|_0^{\pi} = (\pi - 0) - \frac{1}{2}(\sin(2\pi) - \sin(0))$

$= \pi - 0 + 0 - 0 = \boxed{\pi}$. (Again, this was a circle of radius 1, so all good).

③ $r = \theta$ from $0 \le \theta \le \pi$.



We specified the bounds in the problem because it's not a closed curve. So, $A = \int_0^{\pi} \frac{1}{2}(\theta)^2 d\theta = \frac{1}{6}\theta^3\Big|_0^{\pi} = \boxed{\frac{\pi^3}{6}}$.