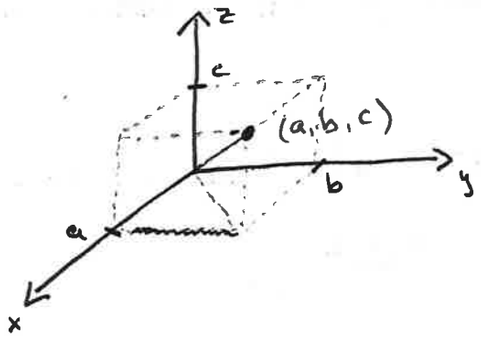
§11.1

Main Concepts

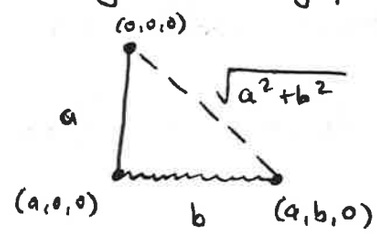
1. Coordinates in 3D
2. Formulae (Distance, Midpoint)
3. Spheres
4. Planes
5. Curves

1. In 2-dimensions, we identify points in the plane by 2 coordinates: x and y. For 3-D, we do the same with an added "z"-coordinate.

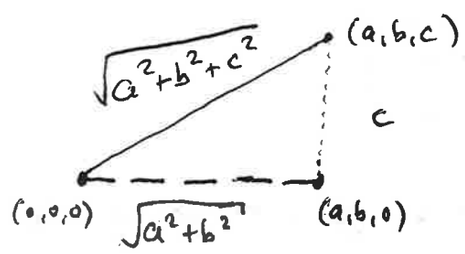


2. This picture also gives us a way to compute distance:

The triangle lying in the xy-plane looks like—



So we find the dashed length by the Pythagorean Theorem. The "standing" triangle looks like—



so the total distance from the origin to (a,b,c) is $\sqrt{a^2+b^2+c^2}$.

Thus, for two points (x_1, y_1, z_1) and (x_2, y_2, z_2), the distance between them is:

DISTANCE FORMULA $d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

This is useful on its own (for computing distance) but will also provide the tool to study other objects/formulae.

Example -

Find the distance between $(1,2,4)$ and $(1,-1,7)$.

Solution: distance $= \sqrt{(1-1)^2+(-1-2)^2+(7-4)^2} = \sqrt{(0)^2+(-3)^2+(3)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$.

If each of the lengths x_2-x_1, y_2-y_1, z_2-z_1 were independently cut in half, the overall distance would also get cut in half. This gives us:

MIDPOINT FORMULA The midpoint of $(x_1,y_1,z_1),(x_2,y_2,z_2)$ has coordinates $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.

3 Another conclusion we can reach from the distance formula is the equation for a sphere. In 2D, a circle is the set of all points (x,y) that are a distance r (for "radius") from the center (a,b). In 3D, the same idea creates a sphere: all points a distance "r" from the center. This gives the equation:

$$\text{distance} = r$$

i.e. $$\sqrt{(x-a)^2+(y-b)^2+(z-c)^2} = r$$

i.e. $$\boxed{(x-a)^2+(y-b)^2+(z-c)^2 = r^2}$$

The equation for a sphere might not be given to you in this form, but if you can put it in this form, you can "read off" the center (a,b,c) and the radius r.

In-class Examples: Identify the center & radius of the following spheres.

① $x^2+(y-1)^2+(z+3)^2=9$

Rewriting this gives: $(x-0)^2+(y-1)^2+(z-(-3))^2=(3)^2$

so the center is $(0,1,-3)$ and the radius is 3.

② $x^2+y^2+z^2-36=0$

Rewriting gives: $(x-0)^2+(y-0)^2+(z-0)^2=6^2$

so the center is $(0,0,0)$ and the radius is 6.

③ $x^2+2x+y^2+z^2=3$

This one looks trickier because of the "2x". But we're looking for $(x-a)^2$ which expands to $x^2-2ax+a^2$. If $-2a=2$ (↳ $a=-1$) then we would have a "2x". So we also want $a^2=(-1)^2=1$. To get it, add it to both sides:

$(x^2+2x+1)+y^2+z^2=3+1$.

Then factor and rewrite:

$(x+1)^2+y^2+z^2=4 \longrightarrow (x-(-1))^2+(y-0)^2+(z-0)^2=2^2$

So the center is $(-1,0,0)$ and the radius is 2.

④ $x^2+10x+y^2-2y+z^2-4z=-5$.

Same idea as ③ but in each coordinate:

$(x^2+10x+25)+(y^2-2y+1)+(z^2-4z+4)=-5+25+1+4$

$(x+5)^2+(y-1)^2+(z-2)^2=25$

So the center is $(-5,1,2)$ and the radius is 5.

4 A plane is a flat 2D surface sitting in space. We have the "typical" examples: the xy-plane (the ground), the yz-plane, and the xz-planes (walls). But they can be tilted as well.

We'll have more to say after we've studied vectors, but for now, the equation of a plane is:

Equation of a Plane $Ax + By + Cz = D$.

We'll finish the discussion (for now) with a suggestion for graphing. A plane is determined by 3 of its points (as long as they don't happen to be on the same straight line). So, to graph a plane, plug in values for 2 of the coordinates and solve for the 3rd. Doing this 3 times will give you the three points.

Examples: Graph the following planes.

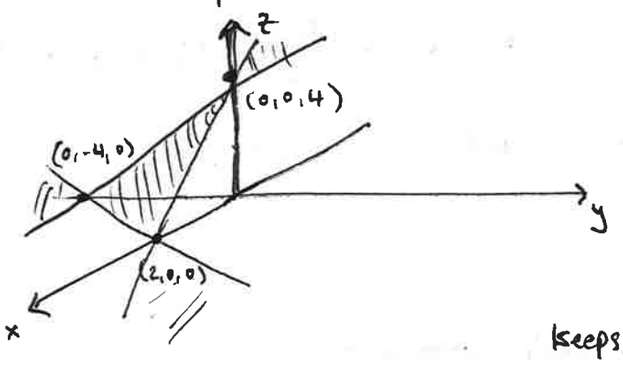
① $2x - y + z = 4$

With $x=y=0$ we get $z=4 \longrightarrow (0,0,4)$ is on the plane.

With $x=z=0$ we get $-y=4$ so $y=-4 \longrightarrow (0,-4,0)$

With $y=z=0$ we get $2x=4$ so $x=2 \longrightarrow (2,0,0)$.

So we join these points:



(Note that the plane is "infinite" in each direction but drawing just the triangle keeps it from getting too messy in my opinion).

② $3x + y = 6$

Note that this is still a plane (with $C=0$).

Setting $x=z=0 \longrightarrow y=6$ $(0,6,0)$

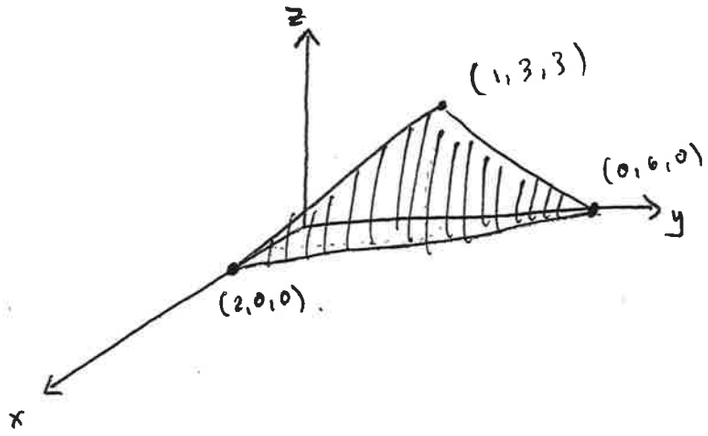
$y=z=0 \longrightarrow 3x=6 \rightarrow x=2$ $(2,0,0)$

But $x=y=0 \longrightarrow 0=6$ so we need different points instead.

Let's try $x=1, z=3 \longrightarrow 3+y=6 \rightarrow y=3$ $(1,3,3)$

(lots of choices will work)

So we graph our 3 points and join them:



(In fact, since there is no "z" in the equation, we can use any z-value. This tells the plane is "vertical" infinitely).

5 Finally, we get to 3D curves. These work the same way as in 2D as well, but need a z-coordinate:

$$x(t), \quad y(t), \quad z(t), \quad t \in I.$$

We can graph, substitute, differentiate, and integrate in the same ways. In particular, the arclength formula generalizes:

ARCLENGTH FORMULA Given a parametrized curve as above, the arclength for $t \in [t_1, t_2]$ is: $$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

Example:

Find the arclength of $x(t) = 4\cos(t)$ $y(t) = 3t$ $z(t) = 4\sin(t)$ $t \in [0, 10]$

Solution: First, $\frac{dx}{dt} = -4\sin(t)$ $\frac{dy}{dt} = 3$ and $\frac{dz}{dt} = 4\cos(t)$. So

$$L = \int_0^{10} \sqrt{(-4\sin(t))^2 + (3)^2 + (4\cos(t))^2}\, dt = \int_0^{10} \sqrt{16\sin^2(t) + 9 + 16\cos^2(t)}\, dt$$

$$= \int_0^{10} \sqrt{16(\sin^2(t) + \cos^2(t)) + 9}\, dt = \int_0^{10} \sqrt{16 + 9}\, dt = \int_0^{10} \sqrt{25}\, dt = \int_0^{10} 5\, dt = 5(10 - 0)$$

$$= 50$$