§11.3

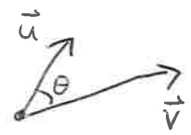
Main Concepts

1. Definition of dot product (geometry + algebra).
2. Orthogonality & Angles
3. Projection
4. Planes
5. Application: Distance from point to plane.

---

1 So far, we don't have a way to multiply vectors together. The dot product gives one such process, taking two vectors and returning a scalar (number).

Geometric definition: Given two vectors $\vec{u}$ and $\vec{v}$ with an angle of θ between them, define $\boxed{\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta}$



We can use this to determine the algebraic definition. For this, suppose we have vectors $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$. Then, using $\hat{i}, \hat{j}, \hat{k}$, we can write

$$\begin{aligned}\vec{u}\cdot\vec{v} &= (u_1\hat{i} + u_2\hat{j} + u_3\hat{k})\cdot(v_1\hat{i} + v_2\hat{j} + v_3\hat{k})\\ &= u_1v_1\,\hat{i}\cdot\hat{i} + u_1v_2\,\hat{i}\cdot\hat{j} + u_1v_3\,\hat{i}\cdot\hat{k} +\\ &\quad + u_2v_1\,\hat{j}\cdot\hat{i} + u_2v_2\,\hat{j}\cdot\hat{j} + u_2v_3\,\hat{j}\cdot\hat{k} +\\ &\quad + u_3v_1\,\hat{k}\cdot\hat{i} + u_3v_2\,\hat{k}\cdot\hat{j} + u_3v_3\,\hat{k}\cdot\hat{k}\end{aligned}$$

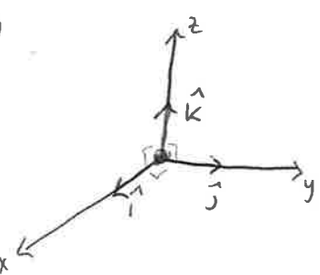
[Distributive Property]

$$= u_1v_1 + u_2v_2 + u_3v_3$$

In this last step, we did 2 things:

1) $\hat{i}\cdot\hat{i} = \|\hat{i}\|\cdot\|\hat{i}\|\cos(0^\circ) = 1\cdot1\cdot1 = 1$ since $\hat{i}$ and itself form a zero degree angle. The same is true for $\hat{j}\cdot\hat{j}$ and $\hat{k}\cdot\hat{k}$.

2) $\hat{i}\cdot\hat{j} = \|\hat{i}\|\cdot\|\hat{j}\|\cos(90^\circ) = 1\cdot1\cdot0 = 0$ since $\hat{i}$ and $\hat{j}$ form a 90° angle. The same is true for $\hat{j}\cdot\hat{k}$ and $\hat{k}\cdot\hat{i}$. So, these other six terms cancel.



Overall, we've learned:

ALGEBRAIC DEFINITION $\langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1v_1 + u_2v_2 + u_3v_3$.

Examples: Compute $\vec{u} \cdot \vec{v}$ given the information provided.

① $\vec{u} = \langle 5, -3, 0 \rangle$ $\quad \vec{v} = \langle 2, 4, -7 \rangle$

From the algebraic definition,

$\vec{u} \cdot \vec{v} = (5)(2) + (-3)(4) + (0)(-7) = 10 - 12 + 0 = -2.$

② $\|\vec{u}\| = 1$, $\|\vec{v}\| = 6$, $\vec{u}$ and $\vec{v}$ form a $30°$ angle.

From the geometric definition,

$\vec{u} \cdot \vec{v} = (1)(6)\cos(30°) = 6 \cdot \frac{\sqrt{3}}{2} = 3\sqrt{3}$.

2 The previous computation also showed us an important fact: if $\vec{u}$ and $\vec{v}$ are perpendicular, then $\vec{u} \cdot \vec{v} = 0$. In fact, the reverse is also true; since $\cos\theta = 0$ only if $\theta = 90°$ (up to a multiple of $180°$).

Even if they're not ~~equal~~ perpendicular, the two forms of the dot product can be used to solve for the angle between them.

Examples: Check if the following pairs of vectors are perpendicular. If not, find the angle between them.

① $\vec{u} = \langle 1, 2, 3 \rangle$ $\quad$ & $\vec{v} = \langle 4, -2, 0 \rangle$

We compute the dot product: $\vec{u} \cdot \vec{v} = (1)(4) + (2)(-2) + (3)(0) = 4 - 4 + 0 = 0$

So $\vec{u}$ and $\vec{v}$ are <u>perpendicular.</u>

② $\vec{u} = \langle 0, 1, 1 \rangle$ and $\vec{v} = \langle -1, 1, 0 \rangle$

The dot product is $\vec{u} \cdot \vec{v} = (0)(-1) + (1)(1) + (1)(0) = 0 + 1 + 0 = 1$ which is not zero.

So, we'll use the geometric formula. First, $\|\vec{u}\| = \sqrt{(0)^2 + (1)^2 + (1)^2} = \sqrt{0+1+1} = \sqrt{2}$ and $\|\vec{v}\| = \sqrt{(-1)^2 + (1)^2 + (0)^2} = \sqrt{1+1+0} = \sqrt{2}$. So:

$$1 = \vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta = (\sqrt{2})(\sqrt{2})\cos\theta = 2\cos\theta.$$

$\longrightarrow \cos\theta = \frac{1}{2}$ $\quad$ so $\quad$ <u>$\theta = 60°$.</u>

③ $\vec{u} = \langle 1, 2, 2 \rangle$ and $\vec{v} = \langle -2, 2, 1 \rangle$.

Then $\vec{u}\cdot\vec{v} = (1)(-2)+(2)(2)+(2)(1) = -2+4+2 = 4$.

This isn't zero, so: $\|\vec{u}\| = \sqrt{(1)^2+(2)^2+(2)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$ and

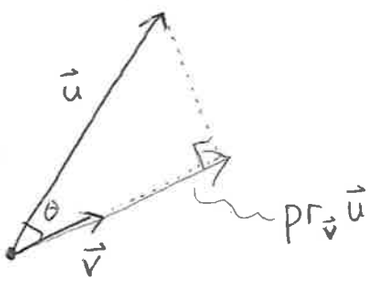
$\|\vec{v}\| = \sqrt{(-2)^2+(2)^2+(1)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$. Then

$$4 = \vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta = (3)(3)\cos\theta = 9\cos\theta. \quad \text{So } \cos\theta = \frac{4}{9}$$

and $\theta = \arccos\left(\frac{4}{9}\right)$.

13 More generally, what we're seeing is that the dot product measures how much two vectors point in the same direction. Indeed, if $\vec{u}$ and $\vec{v}$ have a fixed length, then $\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cos\theta$ is biggest when $\theta = 0$, so $\vec{u}$ and $\vec{v}$ point in exactly the same direction, and it is smallest when $\theta = 180°$, so $\vec{u}$ and $\vec{v}$ point in opposite directions. Finally, we've already seen that $\vec{u}\cdot\vec{v} = 0$ if they are perpendicular, i.e. when they point in completely unrelated directions.

This concept can be summarized by projections. If we have two vectors $\vec{u}$



and $\vec{v}$, the above diagram shows exactly how much $\vec{u}$ points in the $\vec{v}$-direction: this is the vector labeled "$\text{pr}_{\vec{v}}\,\vec{u}$" which we say in English as "the projection of $\vec{u}$ onto $\vec{v}$." Some trigonometry should help us discover an algebraic formula for this vector. First, the magnitude:

$$\frac{\|\text{pr}_{\vec{v}}\,\vec{u}\|}{\|\vec{u}\|} = \cos\theta \quad \left(\frac{\text{adjacent}}{\text{hypotenuse}}\right), \text{ so } \|\text{pr}_{\vec{v}}\,\vec{u}\| = \|\vec{u}\|\cos\theta$$

Then the direction is the same as $\vec{v}$, so we multiply by the unit vector $\hat{v}$:

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

So, overall,

$$\text{pr}_{\vec{v}}\,\vec{u} = \|\text{pr}_{\vec{v}}\,\vec{u}\|\cdot\hat{v} = \|\vec{u}\|\cos\theta\cdot\frac{\vec{v}}{\|\vec{v}\|} = \frac{\|\vec{u}\|\,\|\vec{v}\|\cos\theta}{\|\vec{v}\|^2}\vec{v}$$

(multiply top and bottom by $\|\vec{v}\|$)

$$\boxed{\text{pr}_{\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}}$$

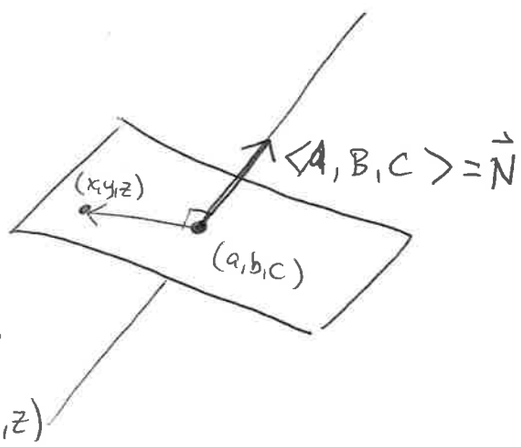
Example: Project $\langle 4,5,6\rangle$ onto the vector $\langle 1,1,0\rangle$.

In other words, for $\vec{u}=\langle 4,5,6\rangle$ and $\vec{v}=\langle 1,1,0\rangle$, we want to compute $pr_{\vec{v}}\vec{u}$. So, $\vec{u}\cdot\vec{v} = (4)(1)+(5)(1)+(6)(0) = 4+5+0 = 9$, and $\|\vec{v}\|^2 = \left(\sqrt{(1)^2+(1)^2+(0)^2}\right)^2 = 1+1+0 = 2$ so

$$pr_{\vec{v}}\vec{u} = \frac{9}{2}\langle 1,1,0\rangle = \left\langle \frac{9}{2}, \frac{9}{2}, 0\right\rangle.$$

4 With the dot product in hand, we can now return to the question of planes. We claimed previously that a plane has equation $Ax+By+Cz=D$. Let us try to "rediscover" this formula and, in so doing, learn more about the coefficients.

In 3D, a plane is completely determined by the line perpendicular to it and one single point on the plane. Suppose the line points in the direction of the vector $\langle A,B,C\rangle$ and contains the point (a,b,c). Then, if (x,y,z) is any point on the plane, we know that $\vec{N}=\langle A,B,C\rangle$ and $\vec{v}$, the vector pointing from (a,b,c) to (x,y,z), are perpendicular. In other words:



$$0 = \vec{N}\cdot\vec{v} = \vec{N}\cdot\left(\langle x,y,z\rangle - \langle a,b,c\rangle\right)$$

so $\vec{N}\cdot\langle x,y,z\rangle = \vec{N}\cdot\langle a,b,c\rangle$.

Plugging in $\vec{N}=\langle A,B,C\rangle$ and $D=\vec{N}\cdot\langle a,b,c\rangle$ gives our usual formula:

$$\langle A,B,C\rangle\cdot\langle x,y,z\rangle = D$$

$$Ax+By+Cz = D.$$

In other words, we've learned not only that our previous equation is correct, but also that the coefficients $\langle A,B,C\rangle$ form a vector that is perpendicular to the whole plane. We call this a normal vector for the plane.

PLANE EQUATION A plane with normal vector $\langle A, B, C \rangle$ has equation

$$Ax + By + Cz = D$$

Examples: For each plane, identify a normal vector.

① $2x - y + z = 7$ ⟶ A normal vector is $\langle 2, -1, 1 \rangle$

② $x + 4z = 1$ ⟶ A normal vector is $\langle 1, 0, 4 \rangle$.

Practice/Review Problem: Identify 3 points on each of the above planes

Example: Find an equation for the plane with normal vector $\langle 1, 2, 3 \rangle$ passing through the point $(6, 5, 4)$.

We know a normal vector, so the plane has equation $1x + 2y + 3z = D$ for some D. The point $(6, 5, 4)$ must satisfy this equation, so

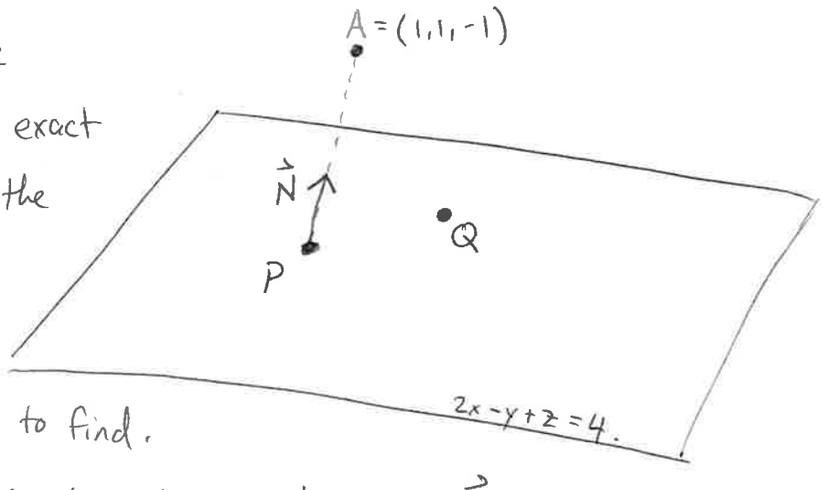
$$(6) + 2(5) + 3(4) = D \longrightarrow D = 6 + 10 + 12 = 28.$$

So, the plane has equation $x + 2y + 3z = 28$.

5 At last, let's put all of our newly-developed skills to solve a problem. As you work through this and similar examples, notice the approach: we draw a good picture, we identify how to solve the problem geometrically, and we use that process to write a sequence of algebraic steps that can be done.

SYNTHESIS PROBLEM: Find the distance from the point $(1, 1, -1)$ to the plane $2x - y + z = 4$.

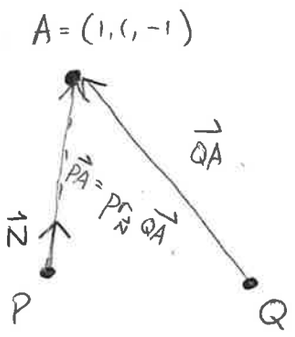
First, let's draw a picture. We're trying to discover a process, so the exact coordinates aren't as important as the general details.



The picture includes the plane, the point, and the distance we're trying to find. Notice that the distance is measured along the normal vector $\vec{N}$.

Unfortunately, we don't know the point P; otherwise we'd use the usual distance formula. We can find some point in the plane (Q)

but it's unlikely to be the one we want.



Nevertheless, let's focus on the three points A, P, Q. From this new picture, we can see that the vector $\vec{PA}$ is the projection of $\vec{QA}$ onto the normal vector $\vec{N}$. But $\|\vec{PA}\|$ is exactly the distance we want to find! So, we have a process:

① Find any point, Q in the plane.

② Calculate $\vec{QA}$

③ Identify $\vec{N}$ from the plane equation

④ Calculate $pr_{\vec{N}} \vec{QA}$.

⑤ The answer is distance $= \|pr_{\vec{N}} \vec{QA}\|$.

So, let's solve our synthesis problem.

① To find a point, pick values for two variables in the plane equation.

Choosing $x = y = 0$ gives $z = 4$ so $Q = (0, 0, 4)$ is a point in the plane.

② $\vec{QA} = \langle 1, 1, -1 \rangle - \langle 0, 0, 4 \rangle = \langle 1, 1, -5 \rangle$.

③ We can read $\vec{N}$ from the plane equation: $\vec{N} = \langle 2, -1, 1 \rangle$.

④ From the projection formula:

$$pr_{\vec{N}} \vec{QA} = \frac{\vec{QA} \cdot \vec{N}}{\|\vec{N}\|^2} \vec{N} = \frac{(1)(2) + (1)(-1) + (-5)(1)}{(2)^2 + (-1)^2 + (1)^2} \langle 2, -1, 1 \rangle$$

$$= \frac{2 - 1 - 5}{4 + 1 + 1} \langle 2, -1, 1 \rangle = -\frac{2}{3} \langle 2, -1, 1 \rangle = \left\langle -\frac{4}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle.$$

⑤ Finally, we get our answer:

$$\text{distance} = \|pr_{\vec{N}} \vec{QA}\| = \sqrt{\left(-\frac{4}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2} = \sqrt{\frac{16}{9} + \frac{4}{9} + \frac{4}{9}}$$

$$= \sqrt{\frac{24}{9}} = \boxed{\frac{2}{3}\sqrt{6}}$$