§11.4

Main Concepts

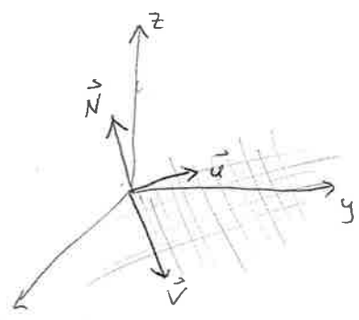
1. Definition of cross product (geometry + algebra).
2. (Anti)commutativity
3. Planes (again).

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1 The dot product represents one way to "multiply" vectors and returns a number. There is another "product" for two vectors in 3D: the cross product.

- Motivation.

Given vectors $\vec{u}$ and $\vec{v}$, if we position them at the origin, they (usually)* define a unique plane:



Question: Can we find a normal vector to this plane?

Answer: Yes, the cross product $\vec{u}\times\vec{v}$.

GEOMETRIC DEFINITION: CROSS PRODUCT The cross product $\vec{u}\times\vec{v}$ of vectors $\vec{u}$ and $\vec{v}$ is a vector satisfying:

① $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\cdot\|\vec{v}\|\sin\theta$ where θ is the angle between $\vec{u}$ and $\vec{v}$.

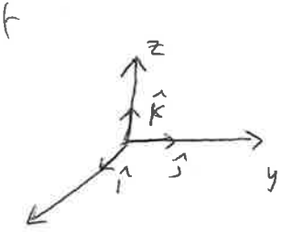
② $\vec{u}\times\vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$.

③ $\vec{u}, \vec{v}, \vec{u}\times\vec{v}$ is a right-handed system.

In other words, ① specifies the magnitude and ② mostly specifies the direction. However, there is still some ambiguity. For example, there are two directions perpendicular to the xy-plane: the positive and negative z-axes. ③ Tells us how to choose the correct one.

In particular, since "xyz" is always right-handed, we get

$\hat{\imath}\times\hat{\jmath} = \hat{k}$, $\hat{\jmath}\times\hat{k} = \hat{\imath}$, and $\hat{k}\times\hat{\imath} = \hat{\jmath}$.



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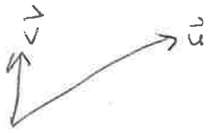
* If two vectors are parallel, they define a line instead of a plane. We'll see in this case that the cross product is zero, which is normal to any plane containing that line.

Examples [Right-hand rule]: For $\vec{u}$, $\vec{v}$ below, decide if $\vec{u}\times\vec{v}$ points "into the page" or "out of the page."

① 

② 

③ 

④ 

(Answers: In, In, In, Out)

Next, we will give an algebraic definition. One way to do this is to do the same calculation we did for the dot product: break up into $\hat{\imath}, \hat{\jmath}, \hat{k}$ components and figure out which terms cancel. Unfortunately six of the nine terms are nonzero, so we'll benefit more from finding a pattern to remember it.

— Determinants —

A matrix is a rectangular grid of numbers (or other objects). To each square matrix, we associate a quantity, the determinant. For a 2×2 matrix, we have

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

2×2 determinant

where the vertical bars tell us to compute the determinant. For a larger matrix, we compute the determinant using smaller submatrices.

Specifically, given a square matrix, the determinant is calculated by taking each entry in the first row, multiplying that number by the determinant of the (smaller) matrix obtained by crossing off its row and column, and adding up these products with alternating plus/minus signs.

Let's do an example:

first row

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 2 & 2 & 5 \end{vmatrix} = +1 \cdot \begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix}$$

alternating signs

Matrix obtained by deleting first row and first column / second column / third column

$$= 1(2\cdot 5 - 3\cdot 2) - 0(1\cdot 5 - 3\cdot 2) + 1(1\cdot 2 - 2\cdot 2) = 4 - 0 - 2 = 2.$$

With this (very brief) crash course on determinants, we can state:

ALGEBRAIC DEFINITION : CROSS PRODUCT Given vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$,

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Examples : Compute $\vec{u} \times \vec{v}$ for each pair $\vec{u}, \vec{v}$:

① $\vec{u} = \langle -9, 5, -3 \rangle \quad \vec{v} = \langle 2, 9, 2 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -9 & 5 & -3 \\ 2 & 9 & 2 \end{vmatrix} = \hat{i}\begin{vmatrix} 5 & -3 \\ 9 & 2 \end{vmatrix} - \hat{j}\begin{vmatrix} -9 & -3 \\ 2 & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} -9 & 5 \\ 2 & 9 \end{vmatrix}$$

$$= \hat{i}(5 \cdot 2 - (-3) \cdot 9) - \hat{j}(-9 \cdot 2 - (-3) \cdot 2) + \hat{k}(-9 \cdot 9 - 5 \cdot 2)$$

$$= \hat{i}(10 + 27) - \hat{j}(-18 + 6) + \hat{k}(-81 - 10) = \langle 37, 12, -91 \rangle$$

② $\vec{u} = \langle -5, 6, 7 \rangle \quad \vec{v} = \langle 1, -9, 0 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5 & 6 & 7 \\ 1 & -9 & 0 \end{vmatrix} = \hat{i}\begin{vmatrix} 6 & 7 \\ -9 & 0 \end{vmatrix} - \hat{j}\begin{vmatrix} -5 & 7 \\ 1 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} -5 & 6 \\ 1 & -9 \end{vmatrix}$$

$$= \hat{i}(6 \cdot 0 - 7 \cdot (-9)) - \hat{j}(-5 \cdot 0 - 7 \cdot 1) + \hat{k}(-5 \cdot (-9) - 6 \cdot 1)$$

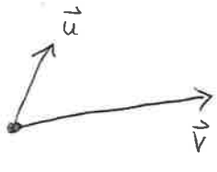
$$= \hat{i}(0 + 63) - \hat{j}(0 - 7) + \hat{k}(45 - 6) = \langle 63, 7, 39 \rangle$$

③ $\vec{u} = \langle 1, 2, 0 \rangle \quad \vec{v} = \langle 0, 5, -1 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ 0 & 5 & -1 \end{vmatrix} = \hat{i}(-2 - 0) - \hat{j}(-1 - 0) + \hat{k}(5 - 0)$$

$$= \langle -2, 1, 5 \rangle$$

2 Notice that geometric property ③ (right-handedness) implies that while $\vec{u} \times \vec{v}$ is into the page, $\vec{v} \times \vec{u}$ is <u>out</u> of the page. So, we see that $\vec{u} \times \vec{v} \neq \vec{v} \times \vec{u}$ in this case. In fact, as we've noted, the only difference is "up vs. down" — not the magnitude or direction line. So, we get that



ANTICOMMUTATIVITY $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

This is especially important to remember while we do calculations, since it means we can't switch the order of terms.

*Exercise: For each of the previous examples, compute $\vec{v} \times \vec{u}$ to check that you indeed get $-(\vec{u} \times \vec{v})$.

3 We now have the tool to write the equation for a plane given 3 of its (noncollinear) points. Indeed, if P, Q, R are points on the plane, then $\vec{PQ}$ and $\vec{QR}$ lie in the plane. So, $\vec{PQ} \times \vec{QR}$ is perpendicular to the plane, i.e. it is a normal vector. So we have the normal vector and a point (say P) and so we can write the equation as we did before.

— Example: Find an equation for the plane containing the points

$P = (5, 2, 6)$ $Q = (1, 0, 0)$ and $R = (-3, 1, -3)$

As suggested, we compute $\vec{QP} = \langle 5, 2, 6 \rangle - \langle 1, 0, 0 \rangle = \langle 4, 2, 6 \rangle$

and $\vec{QR} = \langle -3, 1, -3 \rangle - \langle 1, 0, 0 \rangle = \langle -4, 1, -3 \rangle$.

Then $$\vec{N} = \vec{QP} \times \vec{QR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 2 & 6 \\ -4 & 1 & -3 \end{vmatrix} = \hat{i}(-6-6) - \hat{j}(-12-(-24)) + \hat{k}(4-(-8))$$

$$= \langle -12, -12, 12 \rangle.$$

So the plane has equation $-12x - 12y + 12z = D$ for some D.

Plugging in Q gives: $-12(1) - 12(0) + 12(0) = D \longrightarrow D = -12$

So an equation for the plane is $\boxed{-12x - 12y + 12z = -12}$

(Note: since -12 is common, we can divide it out to get a cleaner equation: $\boxed{x + y - z = 1}$.

This describes the same plane.)