§11.5

Main Concepts

1. Parametrized Curves via vectors
2. Derivatives & tangent vectors
3. Position, velocity, acceleration, speed, arclength.

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1. Recall that in 3D we parameterized curves via 3 functions and bounds:

$$x(t) \qquad y(t) \qquad z(t) \qquad t \in I$$

We can now recognize that this is the same as specifying a vector for each $t \in I$ giving the (x, y, z) coordinates as the components:

VECTOR PARAMETRIZATION OF CURVE A curve is parametrized by a vector-valued function $\vec{r}(t)$ and bounds: $t \in I$.

Example: (This is the same as in §11.1)

$\vec{r}(t) = \langle 4\cos(t), 3t, 4\sin(t) \rangle \qquad t \in [0, 10]$

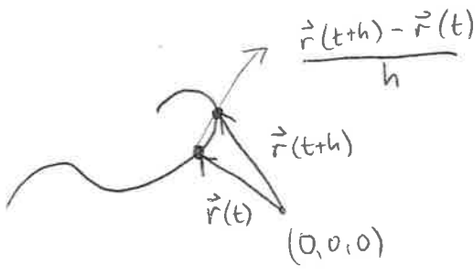
2. Now, we can do calculus on these curves. First, we take derivatives componentwise

VECTOR DERIVATIVE For a space curve parametrized by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, the derivative is $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$.

This settles the algebra, so what about the geometry? We remember the definition of the derivative:

$$\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

Dividing by h only rescales, so to determine the direction, only consider the numerator. Then, from the below picture, we

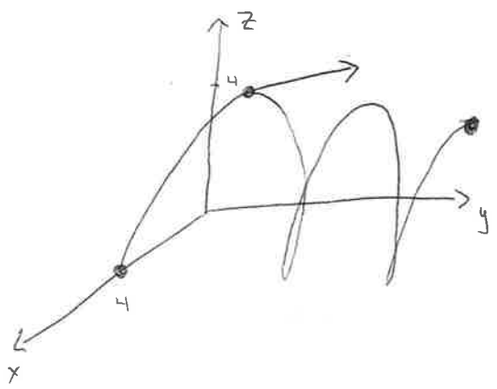


see that $\vec{r}(t+h)-\vec{r}(t)$ looks nearly tangent to the curve. In the limit, this is exactly true:

FACT $\vec{r}\,'(t)$ is tangent to the curve parametrized by $\vec{r}(t)$.

Example: For the previous example, compute $\vec{r}\,'(t)$. Then sketch $\vec{r}(t)$ and $\vec{r}\,'(\frac{\pi}{2})$.

First, $\vec{r}\,'(t) = \langle -4\sin(t),\ 3,\ 4\cos(t)\rangle$. The graph of $\vec{r}(t)$ is a helix of radius 4 in the "y-direction"



Plugging in,

$$\vec{r}\,'(\tfrac{\pi}{2}) = \langle -4\sin(\tfrac{\pi}{2}),\ 3,\ 4\cos(\tfrac{\pi}{2})\rangle$$
$$= \langle -4,\ 3,\ 0\rangle$$

Graphically, this should be tangent to the curve at $\vec{r}(\frac{\pi}{2}) = \langle 4\cos(\frac{\pi}{2}),\ 3(\frac{\pi}{2}),\ 4\sin(\frac{\pi}{2})\rangle = \langle 0,\ \frac{3\pi}{2},\ 4\rangle$ in the direction of motion.

3 Finally, we introduce some language from physics. Given a parametrization $\vec{r}(t)$ of a curve where "t" represents time, we say

- $\vec{r}(t)$ is the position at time t
- $\vec{v}(t) = \vec{r}\,'(t)$ is the velocity at time t.
- $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$ is the acceleration at time t.
- $\|\vec{v}(t)\| = \|\vec{r}\,'(t)\|$ is the speed at time t.

For arclength, this allows us to write the formula somewhat more simply:

$$\text{Arc Length} = \int_{t_0}^{t_1} \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 + \left(\tfrac{dz}{dt}\right)^2}\,dt = \int_{t_0}^{t_1} \|\vec{v}(t)\|\,dt$$

I.e. arc length is the integral of speed.

Example: For the above, compute the velocity, acceleration, speed, and arclength.

We already found $\vec{v}(t) = \vec{r}\,'(t) = \langle -4\sin(t),\ 3,\ 4\cos(t)\rangle$. Then the acceleration

is $\vec{a}(t) = \vec{v}\,'(t) = \langle -4\cos(t),\ 0,\ -4\sin(t) \rangle$. The speed is

$$\|\vec{v}(t)\| = \sqrt{(-4\sin(t))^2 + (3)^2 + (4\cos(t))^2} = \sqrt{16\sin^2(t) + 9 + 16\cos^2(t)} =$$

$$= \sqrt{16(\sin^2(t) + \cos^2(t)) + 9} = \sqrt{16 + 9} = \sqrt{25} = 5$$

Finally the arclength is (remember the bounds $t \in [0, 10]$):

$$L = \int_0^{10} \|\vec{v}(t)\|\, dt = \int_0^{10} 5\, dt = 5(10 - 0) = 50.$$