§11.6

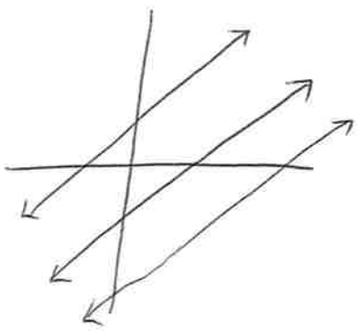
Main Concepts

1. Equation of a line
2. Tangent line to a curve

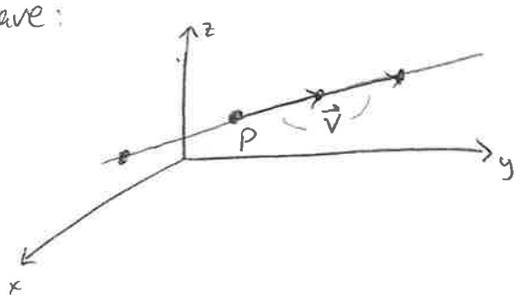
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1 A very specific type of curve that we haven't yet written a parametrization for is a line. We know that a line points in a specific direction, so to write a parametrization, we'll need a direction vector: $\vec{v}$.

But a vector has no position, while a line does (the three lines drawn have the same direction but aren't the same line). So, we also need to specify the position with an initial point P.



Graphically, we have:



We can see that P, $P+\vec{v}$, $P+2\vec{v}$, $P-\vec{v}$, etc. are all points on the line. This gives us our parametrization:

PARAMETRIZATION OF A LINE Given a direction vector $\vec{v}$ and an initial point P, the line passing through P in the $\vec{v}$ direction has parametrization

$$\vec{l}(t) = \vec{P} + t\vec{v}. \qquad t \in \mathbb{R}.$$

Examples: Parametrize the following lines

- Passing through $(1,0,1)$ in the direction of $\langle -1,-1,2 \rangle$.

$$\vec{l}(t) = \langle 1,0,1 \rangle + t\langle -1,-1,2 \rangle = \langle 1,0,1 \rangle + \langle -t,-t,2t \rangle = \langle 1-t,\ -t,\ 1+2t \rangle.$$

- Passing through the origin in the direction of $\langle 2,-3,5 \rangle$

$$\vec{l}(t) = \langle 0,0,0 \rangle + t\langle 2,-3,5 \rangle = \langle 2t,\ -3t,\ 5t \rangle.$$

- The z-axis.

  This also includes the origin, and it points in the $\hat{k}$ direction, so a parametrization is:

  $\vec{l}(t) = \vec{0} + t\hat{k} = \langle 0, 0, t \rangle$.

- The line passing through $(1,1,0)$ and $(0,1,1)$.

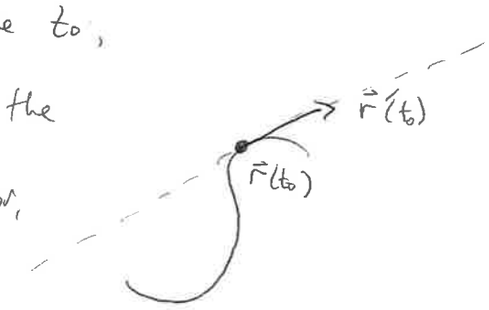
  This hasn't specified a direction. But the vector pointing between these two points is along the line. So $\vec{v} = \langle 1,1,0 \rangle - \langle 0,1,1 \rangle = \langle 1, 0, -1 \rangle$ is a direction vector.

  This gives the parametrization

  $$\vec{l}(t) = \langle 1,1,0 \rangle + t\langle 1,0,-1 \rangle = \langle 1+t, 1, -t \rangle$$

  [You could also use $(0,1,1)$ for the initial point].

2 A special case of this is writing the parametrization for the tangent line to a curve. Given a curve $\vec{r}(t)$ and a specific time t_0, we know that $\vec{r}(t_0)$ lies on the curve and so the tangent line as well. We also have a direction vector, since we know $\vec{r}'(t_0)$ is tangent to the curve. So:



PARAMETRIZATION OF TANGENT LINE The tangent line to $\vec{r}(t)$ at $t = t_0$ has parametrization $\vec{l}(t) = \vec{r}(t_0) + t\vec{r}'(t_0)$.

Example: Find a parametrization for the tangent line to $\vec{r}(t) = \langle 4\cos(t), 3t, 4\sin(t) \rangle$ at $t = \pi$.

We are given $t_0 = \pi$, so $\vec{r}(t_0) = \langle 4\cos(\pi), 3(\pi), 4\sin(\pi) \rangle = \langle -4, 3\pi, 0 \rangle$

We differentiate: $\vec{r}'(t) = \langle -4\sin(t), 3, 4\cos(t) \rangle$, so the tangent vector is: $\vec{r}'(\pi) = \langle -4\sin(\pi), 3, 4\cos(\pi) \rangle = \langle 0, 3, -4 \rangle$.

So, the tangent line has parametrization

$$\vec{l}(t) = \langle -4, 3\pi, 0 \rangle + t\langle 0, 3, -4 \rangle$$