§11.7

Main Concepts

1. Motivating the $\hat{T}, \hat{N}, \hat{B}$ system
2. Defining $\hat{T}, \hat{N}, \hat{B}$.
3. Curvature

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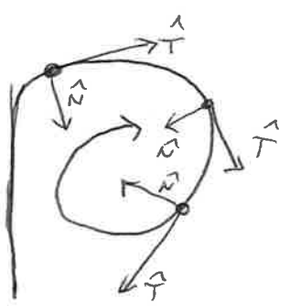
1. Often, when describing 3D space, we use the x-, y-, z-axes (encoded as $\hat{i}, \hat{j}, \hat{k}$) to describe points and vectors. This is often useful and sufficient, but for some applications we may want to choose coordinates relative to the problem at hand.

For the entirety of this section, we will consider an object (a particle) moving through space. Its position will be tracked by $\vec{r}(t)$ at time t. Then, we can choose perpendicular unit vectors describing:

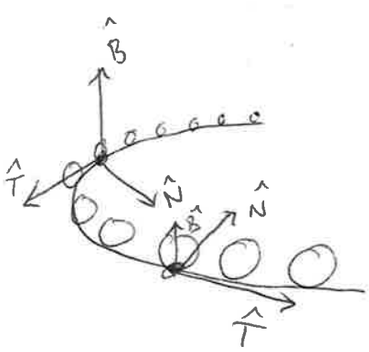
1. Which way is the particle moving? $(\hat{T})$
2. Which way is the particle turning? $(\hat{N})$
3. What is the correct 3rd dimension? $(\hat{B})$.

Example: Let's try to visualize this.

① Suppose $\vec{r}(t)$ looks like so. At the specified points, we've drawn $\hat{T}$ and $\hat{N}$. Notice that they change along the path (so $\hat{T}$ is a function of t; I should really write $\hat{T}(t)$).
For all of these, $\hat{B}$ points into the page.



② For a more 3D example, consider a ball rolling down this ramp (the ball getting larger is supposed to indicate the ramp moving toward us). I've again labelled $\hat{T}$ in the direction of motion, $\hat{N}$ in the direction of turning (left), and $\hat{B}$ now shows up perpendicular to both.



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With the motivation established, let's try to compute. First $\hat{T}$, the unit tangent vector to the curve, is tangent, so it points in the same direction as $\vec{r}\,'(t)$. Making $\vec{r}\,'(t)$ a unit vector gives:

$$\hat{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

Second, let's try to establish a formula for $\hat{N}$, the unit normal vector to the curve. Notice that

$$\hat{T} \cdot \hat{T} = \|\hat{T}\| \cdot \|\hat{T}\| \cos\theta = 1 \cdot 1 \cdot \cos(0) = 1$$

since $\hat{T}$ is a unit vector and $\theta = 0$ between a vector and itself. Differentiating both sides gives:

$$\frac{d}{dt}\left(\hat{T}(t) \cdot \hat{T}(t)\right) = \frac{d}{dt}(1)$$

$$\hat{T}'(t) \cdot \hat{T}(t) + \hat{T}(t) \cdot \hat{T}'(t) = 0 \qquad \text{[Product Rule]}$$

$$2\left(\hat{T} \cdot \hat{T}'\right) = 0$$

$$\hat{T} \cdot \hat{T}' = 0$$

So, $\hat{T}'(t)$ is perpendicular to $\hat{T}(t)$ for all t. This seems to capture the idea we wanted for $\hat{N}$, so we make this a unit vector as well:

$$\hat{N}(t) = \frac{\hat{T}'(t)}{\|\hat{T}'(t)\|}$$

NOTE $\hat{T}(t)$ is a unit vector, but this doesn't mean that $\hat{T}'(t)$ is a unit vector. (Geometrically, moving in a circle of radius 1 doesn't tell you how fast you're going).

Finally, $\hat{B}(t)$ is supposed to be perpendicular to $\hat{T}$ and $\hat{N}$, so we define

$$\hat{B}(t) = \hat{T}(t) \times \hat{N}(t) \qquad \text{[Unit Binormal Vector]}$$

Check: Is $\hat{B}$ really a unit vector?

Yes: $\|\hat{B}\| = \|\hat{T} \times \hat{N}\| = \|\hat{T}\| \, \|\hat{N}\| \sin(90°) = 1 \cdot 1 \cdot 1 = 1$.

- Example: For $\vec{r}(t) = \langle 3\cos(t), 3\sin(t), 2 \rangle$, find $\hat{T}(t)$, $\hat{N}(t)$, and $\hat{B}(t)$.

Following our formulae:

$$\vec{r}\,'(t) = \langle -3\sin(t), 3\cos(t), 0 \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{(-3\sin(t))^2 + (3\cos(t))^2 + (0)^2} = \sqrt{9\sin^2(t) + 9\cos^2(t)} = \sqrt{9(\cos^2(t) + \sin^2(t))} = \sqrt{9} = 3$$

$$\bigstar \quad \hat{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{\langle -3\sin(t), 3\cos(t), 0 \rangle}{3} = \langle -\sin(t), \cos(t), 0 \rangle.$$

$$\hat{T}\,'(t) = \langle -\cos(t), -\sin(t), 0 \rangle$$

$$\|\hat{T}\,'(t)\| = \sqrt{(-\cos(t))^2 + (-\sin(t))^2 + (0)^2} = \sqrt{\cos^2(t) + \sin^2(t)} = \sqrt{1} = 1$$

$$\bigstar \quad \hat{N}(t) = \frac{\hat{T}\,'(t)}{\|\hat{T}\,'(t)\|} = \frac{\langle -\cos(t), -\sin(t), 0 \rangle}{1} = \langle -\cos(t), -\sin(t), 0 \rangle.$$

$$\bigstar \quad \hat{B}(t) = \hat{T}(t) \times \hat{N}(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin(t) & \cos(t) & 0 \\ -\cos(t) & -\sin(t) & 0 \end{vmatrix} = \hat{i}\begin{vmatrix} \cos t & 0 \\ -\sin t & 0 \end{vmatrix} - \hat{j}\begin{vmatrix} -\sin t & 0 \\ -\cos t & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}\left(\sin^2(t) - (-\cos^2(t))\right) = \hat{k}(\sin^2 t + \cos^2 t) = \hat{k}$$

$$= \langle 0, 0, 1 \rangle.$$

Notice, in the end, $\hat{B}$ is constant. This is unsurprising since the particle is confined to a single plane above ($z = 2$) and so $\hat{B}$ is normal to this plane.

Sometimes, we aren't interested in $\hat{B}, \hat{N}, \hat{T}$ for all t, but for a specific t_0. We can always plug in $t = t_0$ into the formulas we get, but we don't need to keep "t" all the way through these calculations. Specifically, we need "t" if we still need to find a derivative.

- Example: For $\vec{r}(t) = \langle t^2, t, \frac{2}{3}t^3 \rangle$, find $\hat{T}(1)$, $\hat{N}(1)$, and $\hat{B}(1)$.

Let's begin again using the general formulae:

$$\vec{r}\,'(t) = \langle 2t, 1, 2t^2 \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{(2t)^2 + (1)^2 + (2t^2)^2} = \sqrt{4t^2 + 1 + 4t^4}$$

$$\hat{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{\langle 2t, 1, 2t^2 \rangle}{\sqrt{1 + 4t^2 + 4t^4}} = \left\langle \frac{2t}{\sqrt{1 + 4t^2 + 4t^4}}, \frac{1}{\sqrt{1 + 4t^2 + 4t^4}}, \frac{2t^2}{\sqrt{1 + 4t^2 + 4t^4}} \right\rangle$$

Notice I could have computed $\hat{T}(1)$ more easily by computing $\vec{r}\,'(1)$ first

so there's less algebra. But since $\hat{N}(t)$ requires $\hat{T}'(t)$ [i.e. I need another derivative], I left things fully general.

So, continuing:

$$\hat{T}'(t) = \Big\langle 2\left(1+4t^2+4t^4\right)^{-1/2} + 2t\left(-\tfrac{1}{2}\right)\left(1+4t^2+4t^4\right)^{-3/2}\left(8t+16t^3\right),\ \left(-\tfrac{1}{2}\right)\left(1+4t^2+4t^4\right)^{-3/2}\left(8t+16t^3\right),$$
$$(4t)\left(1+4t^2+4t^4\right)^{-1/2} + 2t^2\left(-\tfrac{1}{2}\right)\left(1+4t^2+4t^4\right)^{-3/2}\left(8t+16t^3\right)\Big\rangle$$

We could simplify the algebra from here, but since there are no more derivatives needed, let's plug in $t=1$ everywhere:

$$\bigstar\ \hat{T}(1) = \frac{\langle 2(1),\ 1,\ 2(1)^2\rangle}{\sqrt{1+4(1)^2+4(1)^4}} = \frac{\langle 2,\ 1,\ 2\rangle}{\sqrt{1+4+4}} = \frac{\langle 2,1,2\rangle}{\sqrt{9}} = \left\langle \tfrac{2}{3},\ \tfrac{1}{3},\ \tfrac{2}{3}\right\rangle$$

$$\hat{T}'(1) = \Big\langle 2\left(1+4(1)^2+4(1)^4\right)^{-1/2} + 2(1)\left(-\tfrac{1}{2}\right)\left(1+4(1)^2+4(1)^4\right)^{-3/2}\left(8(1)+16(1)^3\right),$$
$$\left(-\tfrac{1}{2}\right)\left(1+4(1)^2+4(1)^4\right)^{-3/2}\left(8(1)+16(1)^3\right),$$
$$4(1)\left(1+4(1)^2+4(1)^4\right)^{-1/2} + 2(1)^2\left(-\tfrac{1}{2}\right)\left(1+4(1)+4(1)^4\right)^{-3/2}\left(8(1)+16(1)^3\right)\Big\rangle$$

$$= \left\langle 2(9)^{-1/2} - (9)^{-3/2}(24),\ -\tfrac{1}{2}(9)^{-3/2}(24),\ 4(9)^{-1/2} - (9)^{-3/2}(24)\right\rangle$$

$$= \left\langle \tfrac{2}{3} - \tfrac{24}{3^3},\ -\tfrac{12}{3^3},\ \tfrac{4}{3} - \tfrac{24}{3^3}\right\rangle$$

$$= \tfrac{1}{27}\langle 18-24,\ -12,\ 36-24\rangle$$

$$= \tfrac{1}{27}\langle -6,\ -12,\ 12\rangle = \tfrac{2}{9}\langle -1,\ -2,\ 2\rangle$$

$$\|\hat{T}'(1)\| = \tfrac{2}{9}\sqrt{(-1)^2+(-2)^2+2^2} = \tfrac{2}{9}\sqrt{1+4+4} = \tfrac{2}{9}\sqrt{9} = \tfrac{2}{3}.$$

$$\bigstar\ \hat{N}(1) = \frac{\hat{T}'(1)}{\|\hat{T}'(1)\|} = \frac{\frac{2}{9}\langle -1,\ -2,\ 2\rangle}{2/3} = \tfrac{1}{3}\langle -1,-2,2\rangle = \left\langle -\tfrac{1}{3},\ -\tfrac{2}{3},\ \tfrac{2}{3}\right\rangle.$$

$$\bigstar\ \hat{B}(1) = \hat{T}(1)\times\hat{N}(1) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2/3 & 1/3 & 2/3 \\ -1/3 & -2/3 & 2/3 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 1/3 & 2/3 \\ -2/3 & 2/3 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 2/3 & 2/3 \\ -1/3 & 2/3 \end{vmatrix} + \hat{k}\begin{vmatrix} 2/3 & 1/3 \\ -1/3 & -2/3 \end{vmatrix}$$

$$= \hat{\imath}\left(\tfrac{2}{9} - \left(-\tfrac{4}{9}\right)\right) - \hat{\jmath}\left(\tfrac{4}{9} - \left(-\tfrac{2}{9}\right)\right) + \hat{k}\left(-\tfrac{4}{9} - \left(-\tfrac{1}{9}\right)\right)$$

$$= \hat{\imath}\left(\tfrac{6}{9}\right) - \hat{\jmath}\left(\tfrac{6}{9}\right) + \hat{k}\left(-\tfrac{3}{9}\right) = \left\langle \tfrac{2}{3},\ -\tfrac{2}{3},\ -\tfrac{1}{3}\right\rangle$$

3 Finally, we define curvature. Curvature measures how much a path is bending: a straight line should have zero curvature, a gentle turn should have small curvature, and a sharp turn should have large curvature.

To compute the curvature, we are interested in the rate at which the direction of travel is changing. In other words, we should consider $\|\hat{T}'(t)\|$. But also, if two particles are moving along the same path at different speeds, then $\|\hat{T}'(t)\|$ will be larger for the faster one. So, we want to correct for this. Overall, this suggests:

CURVATURE FORMULA $\kappa(t) = \dfrac{\|\hat{T}'(t)\|}{\|\vec{r}\,'(t)\|}$.

Examples: ~~Compute $\kappa(t)$ for~~

① Compute $\kappa(t)$ for the first example above.

Directly: $\kappa(t) = \dfrac{\|\hat{T}'(t)\|}{\|\vec{r}\,'(t)\|} = \dfrac{1}{3}$

(In fact, more generally a circle of radius r has curvature $1/r$).

② Compute $\kappa(1)$ for the second example above.

Here, $\kappa(1) = \dfrac{\|\hat{T}'(1)\|}{\|\vec{r}\,'(1)\|} = \dfrac{2/3}{\sqrt{1+4(1)^2+4(1)^4}} = \dfrac{2/3}{\sqrt{9}} = \dfrac{2/3}{3} = \dfrac{2}{9}$.

③ Compute the curvature of a straight line.

First, we need a parametrization. A line is parametrized by

$\vec{r}(t) = \vec{P} + t\vec{v}$. So:

$\vec{r}\,'(t) = \vec{v}$ $\qquad \|\vec{r}\,'(t)\| = \|\vec{v}\|$.

$\hat{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \dfrac{\vec{v}}{\|\vec{v}\|} = \hat{v}$.

$\hat{T}'(t) = \vec{0}$ So $\kappa(t) = \dfrac{\|\hat{T}'(t)\|}{\|\vec{r}\,'(t)\|} = \dfrac{0}{\|\vec{v}\|} = 0$.

as expected.