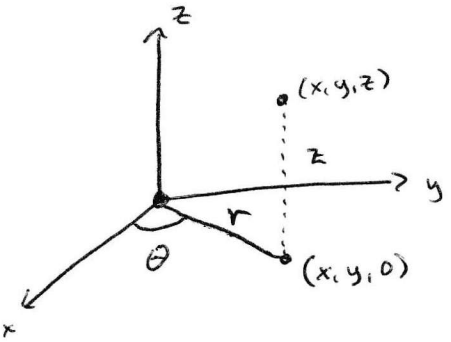
§11.9

Main Concepts

1. Cylindrical Coordinates
2. Spherical Coordinates

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1 So far, we've identified points in space by 3 coordinates: x, y, and z. When we're studying a system that has an axis of symmetry, we have another coordinate system that is convenient: cylindrical coordinates. We'll use cylindrical coordinates with the z-axis as the axis of symmetry. Then, to a point (x,y,z), we associate the quantities r, θ as in the diagram. Namely, r is the distance from the origin to $(x,y,0)$, and θ is the angle between the positive x-axis and the vector $\langle x,y,0\rangle$. Then the point can be identified by the numbers (r, θ, z) and we call these the cylindrical coordinates of (x,y,z). Note: we usually have $r \geq 0$ and $0 \leq \theta < 2\pi$.



Now we can convert back-and-forth between cylindrical and Euclidean coordinates:

Cyl. → Euc.

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

Euc. → Cyl.

$$r = \sqrt{x^2+y^2}$$

$$\theta = \arctan(y/x) \quad [\text{Possibly } \pm\pi]$$

$$z = z.$$

Examples: Convert these points between Euclidean and cylindrical coordinates.

① $(r, \theta, z) = (2, \pi/3, 7)$.

Using the formulae: $x = r\cos\theta = 2\cdot\cos(\pi/3) = 2(\frac{1}{2}) = 1$

$y = r\sin\theta = 2\cdot\sin(\pi/3) = 2(\frac{\sqrt{3}}{2}) = \sqrt{3}$

$z = z = 7$

$\boxed{(x, y, z) = (1, \sqrt{3}, 7)}$

② $(x, y, z) = (2, 2, 3)$.

Again, the formulae give: $z = z = 3$

$r = \sqrt{x^2+y^2} = \sqrt{2^2+2^2} = \sqrt{4+4} = \sqrt{8}$

$\theta = \arctan(y/x) = \arctan(2/2) = \arctan(1) = \pi/4$.

Notice that $(x, y) = (2, 2)$ is in Quadrant I, so $\theta = \pi/4$ is correct.

So, $(r, \theta, z) = (\sqrt{8}, \pi/4, 3)$.

③ $(x, y, z) = (-1, -1, 5)$.

Again: $r = \sqrt{x^2+y^2} = \sqrt{(-1)^2+(-1)^2} = \sqrt{1+1} = \sqrt{2}$.

$\theta = \arctan(\frac{y}{x}) = \arctan(\frac{-1}{-1}) = \arctan(1) = \pi/4$.

$z = z = 5$.

This time, $(x, y) = (-1, -1)$ is in Quadrant III, so we fix θ by adding π. I.e.

$\theta = \pi/4 + \pi = \frac{5\pi}{4}$. So, $(r, \theta, z) = (\sqrt{2}, \frac{5\pi}{4}, 5)$.

We can also use the same formulae to convert equations of surfaces between coordinate systems.

Examples: Convert these surface equations between Euclidean and ~~spherical~~ cylindrical coordinates.

① $x^2+y^2+z^2 = 1$.

Substituting gives: $(r\cos\theta)^2 + (r\sin\theta)^2 + (z)^2 = 1$

$r^2\cos^2\theta + r^2\sin^2\theta + z^2 = 1$

$$r^2(\cos^2\theta + \sin^2\theta) + z^2 = 1$$

$$\boxed{r^2 + z^2 = 1.}$$

(2) $z^2 = \frac{x^2}{4} + \frac{y^2}{4}$

Substitute again: $z^2 = \frac{r^2\cos^2\theta}{4} + \frac{r^2\sin^2\theta}{4}$

$$z^2 = \frac{r^2}{4}(\cos^2\theta + \sin^2\theta)$$

$$z^2 = \frac{r^2}{4}$$

$$\boxed{z = \pm\frac{r}{2}}$$

(3) ~~$x^2 + y^2 = 1$~~ $r = 3\cos\theta$

~~Substitute to get~~ Substituting would give $\cos(\arctan(y/x))$, which is messy.

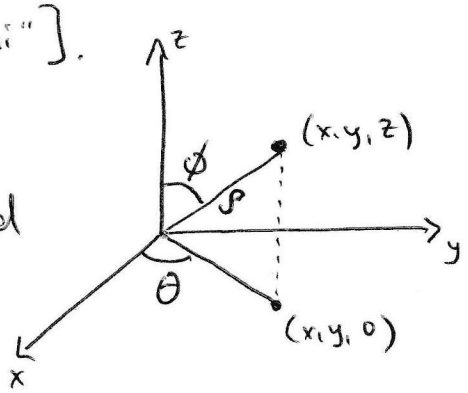
Instead, multiplying by r throughout gives another approach:

$$r^2 = 3r\cos\theta$$

$(x^2+y^2) = 3(x)$, i.e $x^2 - 3x + y^2 = 0$ is a cylinder.

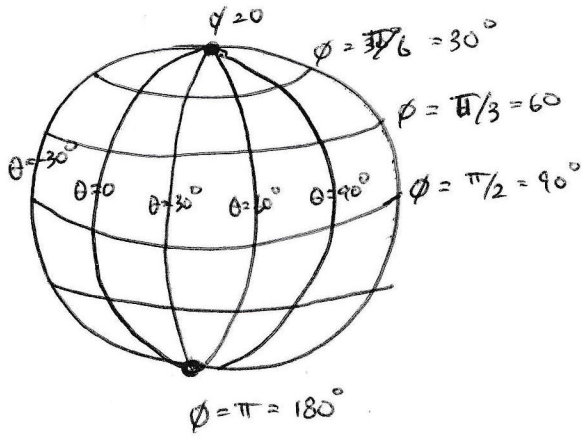
2 If, instead of an axis of symmetry, we have point symmetry (say, for example, thinking about space around the nucleus of an atom), then we may benefit from spherical coordinates. We now identify a point (x,y,z) by 3 numbers (ρ, θ, ϕ) [this last letter is pronounced "phi"].

ρ is the distance from (x,y,z) to the origin; θ is the same angle as before; and ϕ is the angle between the positive z-axis and $\langle x,y,z\rangle$.



Typically, we take $\rho \geq 0$, $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$

Another useful interpretation of θ, ϕ is used in geography: θ measure longitude while ϕ measures latitude. In this interpretation, ρ is the radius of the sphere that we're sitting on.



As before, we get conversion formulae:

Euc $\longrightarrow$ Sph.

$$\rho = \sqrt{x^2+y^2+z^2}$$

$$\theta = \arctan(y/x) \quad [\text{Possibly } +\pi]$$

$$\phi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \arccos\left(\frac{z}{\rho}\right)$$

Sph. $\longrightarrow$ Euc

$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

And as before, we can convert both points and equations.

Examples: Convert between Euclidean and spherical coordinates.

① $(\rho, \theta, \phi) = (4, 0, \pi/6)$.

Directly: $x = \rho \sin\phi \cos\theta = (4)\sin(\pi/6)\cos(0) = 4 \cdot \frac{1}{2} \cdot 1 = 2$

$y = \rho \sin\phi \sin\theta = (4)\sin(\pi/6)\sin(0) = 4 \cdot \frac{1}{2} \cdot 0 = 0$

$z = \rho \cos\phi = (4)\cos(\pi/6) = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$.

So $(x, y, z) = (2, 0, 2\sqrt{3})$.

② $(x, y, z) = (0, 3, 4)$.

Then $\rho = \sqrt{0^2+3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$.

$\theta = \arctan(3/0)$ (!)

$\phi = \arccos\left(\frac{z}{\rho}\right) = \arccos\left(\frac{4}{5}\right)$.

Notice θ couldn't be solved this way. Using the equation for x gives:

$x = \rho \sin\phi \cos\theta$

$0 = (5)\sin(\arccos(4/5))\cos\theta$

$0 = \cos\theta \longrightarrow$ so $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

Since $(x,y) = (0,3)$ is in Quadrants I and II, we get $\theta = \pi/2$.

So, $(\rho, \theta, \phi) = \left(5, \pi/2, \arccos(4/5)\right)$.

③ $x^2+y^2+z^2 = 9$.

Directly: $(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 + (\rho\cos\phi)^2 = 9$

$\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi = 9$

$\rho^2\sin^2\phi[\cos^2\theta + \sin^2\theta] + \rho^2\cos^2\phi = 9$

$\rho^2\sin^2\phi + \rho^2\cos^2\phi = 9$

$\rho^2(\sin^2\phi + \cos^2\phi) = 9$

$\rho^2 = 9$

$\boxed{\rho = 3}$

Notice a sphere has a simple equation in spherical coordinates.

④ $\rho = 4\cos\phi$.

Again, substituting for ϕ is messy, so it's better to multiply by ρ:

$\rho^2 = 4\rho\cos\phi$

$x^2+y^2+z^2 = 4z$

$x^2+y^2+(z^2-4z+4) = 4$

$x^2+y^2+(z-2)^2 = 4$ This is a sphere centered at $(0,0,2)$ of radius 2.