§12.2

Main Concepts

1. Partial derivatives
2. More variables
3. More derivatives

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1 Now that we can study functions of multiple variables, we can do calculus on them. To begin, we'd like to think about derivatives, i.e. rates of change. But there's an issue: in 1D, we only had to ask what happens when x goes up/down; for $f(x,y)$, the value of the function might increase in some directions and decrease in others.

To resolve this, we start by "isolating" our variables. If we have a function $f(x,y)$, then we can ask what happens when x varies and y is held constant. This gives the partial derivative of f with respect to x, written $f_x(x,y)$ or $\frac{\partial f}{\partial x}$. We can then do the same for y: $f_y = \frac{\partial f}{\partial y}$ is the partial derivative of f with respect to y.

Examples: Compute f_x and f_y for each function $f(x,y)$:

① $f(x,y) = x^2 + y^3$

For f_x, we think of x as a variable and y as constant:

$f_x(x,y) = 2x + 0 = 2x$.

For f_y, we do the opposite:

$f_y(x,y) = 0 + 3y^2 = 3y^2$

② $f(x,y) = xy^2$

Same idea:

$f_x(x,y) = (1)y^2 = y^2$

$f_y(x,y) = x(2y) = 2xy$

⟷ Notice $\frac{\partial}{\partial x}(xy^2) = y^2$. This is for the same reason that $\frac{d}{dx}(7x) = 7$: remember we treat y (and so y^2) as constant.

③ $f(x,y) = x\cos(x+y^2)$

For f_x, we need the product rule:

$f_x(x,y) = \cos(x+y^2) + x(-\sin(x+y^2))\cdot(1) = \cos(x+y^2) - x\sin(x+y^2)$.

$f_y(x,y) = -x\sin(x+y^2)(0+2y) = -2xy\sin(x+y^2)$.

2 The same idea applies to more variables. For $f(x,y,z)$, f_x is the derivative of f when we keep both y and z constant, and similarly for f_y and f_z.

Example: Find f_x, f_y, and f_z for $f(x,y,z) = x^2 y e^{y+z}$.

For f_x, only x^2 is variable:

$f_x = 2xye^{y+z}$

For f_y, we need a product rule:

$f_y = x^2(1)e^{y+z} + x^2 y e^{y+z}\cdot(1+0) = x^2 e^{y+z} + x^2 y e^{y+z}$.

For f_z, only e^{y+z} is variable:

$f_z = x^2 y e^{y+z}\cdot(0+1) = x^2 y e^{y+z}$

3 Taking partial derivatives gives us back a function, so we can again take derivatives. For example, we can take 2 x-derivatives: $(f_x)_x = f_{xx}$ or even mixed partials like $(f_x)_y = f_{xy}$, which is taking the x-derivative and then the y-derivative. NOTE: the notation is backwards for Leibniz ($\frac{\partial}{\partial x}$) notation; f_{xy} would be written: $\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f\right) = \frac{\partial^2 f}{\partial y\,\partial x}$.

Example: Find all second partial derivatives of $f(x,y,z) = x^2 y\sin(z)$.

First, let's find the first derivatives:

$f_x(x,y,z) = 2xy\sin(z)$, $f_y(x,y,z) = x^2\sin(z)$, $f_z(x,y,z) = x^2 y\cos(z)$.

Then we take all derivatives of each of these:

$f_{xx} = 2y\sin(z)$ $\quad$ $f_{xy} = 2x\sin(z)$ $\quad$ $f_{xz} = 2xy\cos(z)$

$f_{yx} = 2x\sin(z)$ $\quad$ $f_{yy} = 0$ $\quad$ $f_{yz} = x^2\cos(z)$

$f_{zx} = 2xy\cos(z)$ $\quad$ $f_{zy} = x^2\cos(z)$ $\quad$ $f_{zz} = -x^2 y\sin(z)$