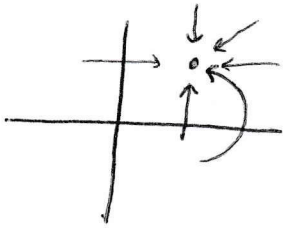
§12.3

Main Concepts

1. Limits
2. Strategies for solving limits
3. Continuity

---

1 In multiple dimensions, limits have added complexity as well, since we can approach a point (a,b) not just from the left and right, but also from above, below, along an angle, or even along a curve. We'll say that $\lim_{(x,y)\to(a,b)} f(x,y) = L$ if $f(x,y)$ approaches L whenever (x,y) approaches (a,b) along **any** path.



2 We'll develop some strategies for solving limits through examples, and then summarize.

Examples: Compute the following limits:

① $\lim_{(x,y)\to(3,4)} (x^2 + xy - y^2)$

As $(x,y)\to(3,4)$, $x\to 3$ and $y\to 4$. So $x^2 \to 3^2 = 9$, $xy \to (3)(4) = 12$, and $y^2 \to 4^2 = 16$.

So $\lim_{(x,y)\to(3,4)} (x^2 + xy - y^2) = (3)^2 + (3)(4) - (4)^2 = 9 + 12 - 16 = \boxed{5}$

② $\lim_{(x,y)\to(7,6)} \dfrac{x^2+1}{y-6}$

The numerator approaches $7^2+1 = 49+1 = 50$. The denominator approaches $6-6=0$. So, this limit doesn't exist; it's getting larger and larger as $y\to 6$ (and $y>6$). **DNE**

③ $\lim_{(x,y)\to(1,2)} \dfrac{2x^2+xy-y^2}{2x-y}$

Plugging in again gives $2x^2+xy-y^2 \to 2(1)^2+(1)(2)-(2)^2 = 2+2-4=0$ and

$2x-y \longrightarrow 2(1)-(2) = 2-2=0$

So, this has the indeterminate form $\frac{0}{0}$. As in the single-variable case, this

means we have a chance of evaluating it. We don't have a L'Hôpital's rule (we can't even take 2-variable derivatives yet), so we use our other approach: factoring.

$$\lim_{(x,y)\to(1,2)} \frac{2x^2+xy-y^2}{2x-y} = \lim_{(x,y)\to(1,2)} \frac{(2x-y)(x+y)}{2x-y} = \lim_{(x,y)\to(1,2)} (x+y) = (1)+(2) = \boxed{3}$$

(4) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

Plugging in gives $\frac{(0)^2-(0)^2}{(0)^2+(0)^2} = \frac{0}{0}$, the indeterminate form. We could again try factoring, but this won't work since x^2+y^2 can't be factored and we certainly won't get any cancellation.

We might suspect that the limit doesn't exist now, so let's investigate different paths.

Approaching $(0,0)$ horizontally is along the line $y=0$, so we get:

$$\lim_{x\to 0} \frac{x^2-(0)^2}{x^2+(0)^2} = \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0}(1) = 1.$$

We're not done; we've only checked one path. Let's try a vertical path ($x=0$):

$$\lim_{y\to 0} \frac{(0)^2-y^2}{(0)^2+y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = \lim_{y\to 0}(-1) = -1.$$

The limit can't have different values on different paths, so the limit $\boxed{\text{DNE}}$.

(5) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

We'll skip the initial discussion: this is $\frac{0}{0}$ and can't be factored usefully. Horizontally ($y=0$):

$$\lim_{x\to 0} \frac{x(0)}{x^2+(0)^2} = \lim_{x\to 0}(0) = 0$$



and vertically ($x=0$):

$$\lim_{y\to 0} \frac{(0)y}{(0)^2+y^2} = \lim_{y\to 0}(0) = 0$$

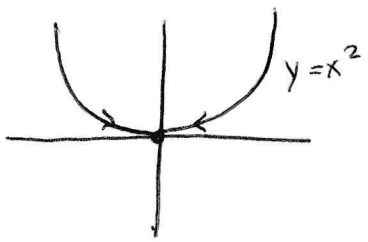
give the same. But there are other paths. Approaching diagonally (along $y=x$) gives:

$$\lim_{x\to 0} \frac{x(x)}{x^2+(x)^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$$

But $0 \neq 1/2$, so the limit $\boxed{\text{DNE}}$

(6) $\lim_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$.

Again, plugging in and factoring don't help. Trying paths:



$y=0$: $\lim_{x\to0} \dfrac{x^2(0)}{x^4+(0)^2} = 0$

$x=0$: $\lim_{y\to0} \dfrac{(0)^2y}{(0)^4+y^2} = 0$

$y=x$: $\lim_{x\to0} \dfrac{x^2(x)}{x^4+(x)^2} = \lim_{x\to0} \dfrac{x^3}{x^4+x^2} = \lim_{x\to0} \dfrac{x}{x^2+1} = \dfrac{(0)}{(0)^2+1} = \dfrac{0}{1} = 0$.

So far, every path has given the same. Let's try any line through the origin $y=mx$ (with nonzero slope $m\neq0$ since $y=0$ we already tried). Then:

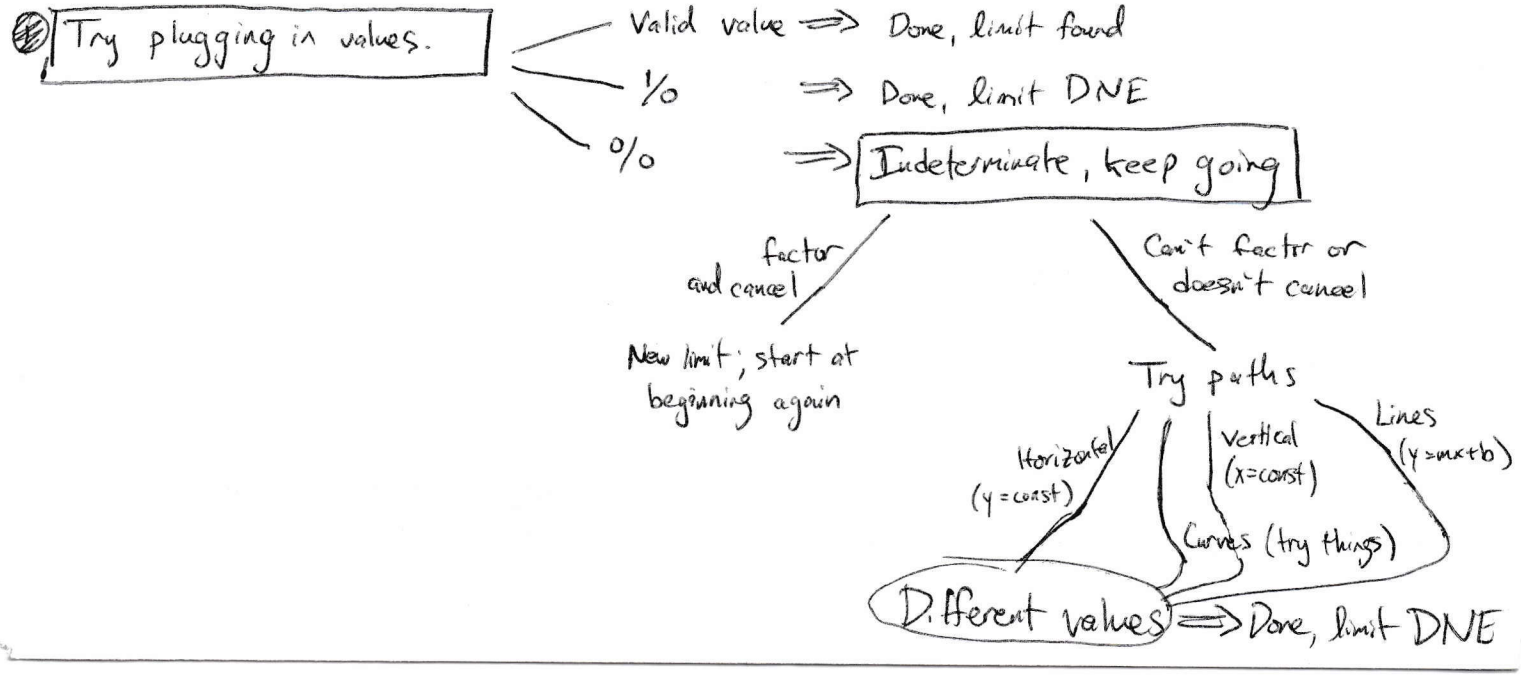
$y=mx$: $\lim_{x\to0} \dfrac{x^2(mx)}{x^4+(mx)^2} = \lim_{x\to0} \dfrac{mx^3}{x^4+m^2x^2} = \lim_{x\to0} \dfrac{mx}{x^2+m^2} = \dfrac{m(0)}{(0)^2+m^2} = \dfrac{0}{m^2} = 0$

since $m\neq0$. We've now tried every straight line path and gotten the same value. But there are curved paths as well, such as the path $y=x^2$ above.

$y=x^2$: $\lim_{x\to0} \dfrac{x^2(x^2)}{x^4+(x^2)^2} = \lim_{x\to0} \dfrac{x^4}{2x^4} = \lim_{x\to0} \dfrac{1}{2} = \dfrac{1}{2}$. But $\frac{1}{2}\neq0$.

And so, at last, we conclude that the limit **DNE**.

Finally, we summarize our strategy:



[3] Finally, a note. When plugging in works $\left(\text{i.e. } \lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)\right)$ we say that f is continuous at (a,b). For most of the functions we will see, they will be continuous wherever they are defined. So, for example $f(x,y) = x^2+xy-y^3$ is continuous everywhere; $f(x,y) = \sqrt{x-y}$ is continuous for $x-y \geq 0$ $(x \geq y)$; and $f(x,y) = \log(x^2+y)$ is continuous for $x^2+y > 0$ $(x^2 > -y)$.

This definition is useful because it explains a phenomena we saw in §12.2.

Theorem (Clairaut): If f_{xy} and f_{yx} are continuous, then $f_{xy} = f_{yx}$.

Or, another version,

Theorem: If f_{xy} and f_{yx} exist and f_{xy} is continuous, then $f_{xy} = f_{yx}$.