§12.4

Main Concepts

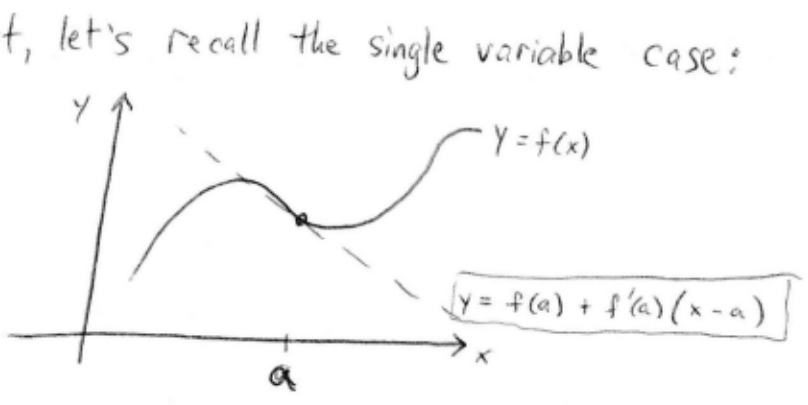
[1] Tangent planes & differentiability

[2] The gradient

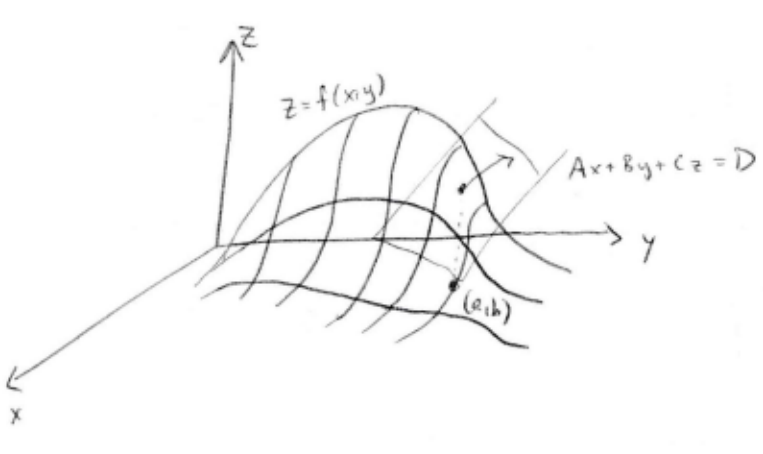
[3] Tangent planes, again

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[1] Now we can try to establish what a "total derivative" of a multivariable function should be. To start, let's recall the single variable case:



To find $f'(a)$, we want to approximate the curve with a line. This gives us the equation above, and $f'(a)$ is the slope as usual. With this picture in mind, we can ask – how can we approximate $f(x,y)$ with something flat?



Here, we see a surface is approximated well by a tangent plane. Solving the equation for a plane for z gives: $z = \frac{D}{C} - \frac{A}{C}x - \frac{B}{C}y$, or after rewriting with new constants: $z = k + r(x-a) + s(y-b)$, and at (a,b) we know $z = f(a,b)$. So, we get $\boxed{z = f(a,b) + \vec{\nabla} f(a,b) \cdot \langle x-a, y-b \rangle}$.

where $\vec{\nabla} f$ plays the role of the derivative (compare with the boxed equation above).

NOTE: For some functions, there is not a good approximation by a plane. We say a function is differentiable if it can be approximated by a plane.

We call $\vec{\nabla} f$ the gradient of f, or simply grad f for short.

[2] With the theory set up, we should ask how to compute $\vec{\nabla} f$. From what we wrote, the approach should be to find the plane equation and solve for z.

To do this, note that if I move 1 unit in the "x" direction, then the function increases by about f_x (thinking of f_x = derivative = slope = $\frac{\text{rise}}{\text{run}}$). So, $\langle 1, 0, f_x \rangle$ lies in the plane. By the same reasoning, $\langle 0, 1, f_y \rangle$ lies in the plane. So, a normal vector to the plane is:

$$\langle 0,1,f_y \rangle \times \langle 1,0,f_x \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & f_y \\ 1 & 0 & f_x \end{vmatrix} = \langle f_x - 0, -(0 - f_y), 0 - 1 \rangle$$

$$= \langle f_x, f_y, -1 \rangle.$$

So the tangent plane is $(f_x)x + (f_y)y + (-1)z = D$. Plugging in the point $(a,b,f(a,b))$ gives $D = af_x + bf_y - f(a,b)$, so we have

$$f_x x + f_y y - z = af_x + bf_y - f(a,b) \quad \text{or:}$$

$$z = f_x x + f_y y - af_x - bf_y + f(a,b) = f(a,b) + f_x(x-a) + f_y(y-b).$$

$$= f(a,b) + \langle f_x, f_y \rangle \cdot \langle x-a, y-b \rangle.$$

GRADIENT, TWO VARIABLES For a function $f(x,y)$, $\vec{\nabla} f = \langle f_x, f_y \rangle$.

The same definition is useful for any number of variables. In particular

GRADIENT, THREE VARIABLES For a function $f(x,y,z)$, $\vec{\nabla} f = \langle f_x, f_y, f_z \rangle$.

Examples: For each function f, compute $\vec{\nabla} f$.

① $f(x,y) = x^2y + 3xy$

$$\vec{\nabla} f = \langle f_x, f_y \rangle = \langle 2xy + 3y, x^2 + 3x \rangle$$

② $f(x,y) = x\log(x+y^2)$

$$\vec{\nabla} f = \langle f_x, f_y \rangle = \left\langle \log(x+y^2) + x\left(\tfrac{1}{x+y^2}\right)\cdot 1, x\left(\tfrac{1}{x+y^2}\right)(2y) \right\rangle$$

$$= \left\langle \log(x+y^2) + \frac{x}{x+y^2}, \frac{2xy}{x+y^2} \right\rangle.$$

[Notice the product & chain rules].

③ $f(x,y,z) = xyz\sin(x)$

$$\vec{\nabla} f = \langle f_x, f_y, f_z \rangle = \langle yz\sin(x) + xyz\cos(x), xz\sin(x), xy\sin(x) \rangle.$$

[3] Finally, another note on tangent planes. This is a practical note as I expect many of you will come here to find examples. The approach used in the textbook in this section is unsatisfying to me; instead, I would recommend using the approach in §12.7 (so open those notes). I'll start one example here to demonstrate:

(Partial) Example: Find an equation for the tangent plane to $f(x,y) = x^2y$ at the point $(1,2)$.

To begin, we write $z = f(x,y)$, so $z = x^2y$. Then, to use the approach in §12.7, we want all of our variables on one side, so we get $0 = x^2y - z$. We also need a point on the surface, so plug in $(1,2)$ to find z:

$z = (1)^2(2) = 2$, so $(1,2,2)$ is on the surface.

Now use §12.7 [$F(x,y,z) = x^2y - z$ and the point $(1,2,2)$].