§12.5

Main Concepts

1. Directional derivatives
2. Interpretation of gradient
3. Visualization

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1. The gradient is the last tool we needed to answer the question: how fast does my function change as I move in a specific direction (not just along the xyz axes).

We'll write $D_{\hat{u}} f$ for the directional derivative of f in the $\hat{u}$ direction.

Notice that $\hat{u}$ is a unit vector, since we're interested in a direction only. We can compute it using our usual definition of the derivative:

$$D_{\hat{u}} f(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\hat{u}) - f(\vec{a})}{h}$$

$$= \lim_{h \to 0} \frac{\left[f(\vec{a}) + \vec{\nabla} f(\vec{a}) \cdot ((\vec{a} + h\hat{u}) - \vec{a}) \right] - f(\vec{a})}{h}$$

using the tangent plane approximation from §12.4

$$= \lim_{h \to 0} \frac{\vec{\nabla} f(\vec{a}) \cdot (h\hat{u})}{h} = \lim_{h \to 0} \vec{\nabla} f(\vec{a}) \cdot \hat{u} = \vec{\nabla} f(\vec{a}) \cdot \hat{u}.$$

That is, $\boxed{D_{\hat{u}} f = \vec{\nabla} f \cdot \hat{u}}$

Examples: Find the rate of change of $f(x,y) = x^2 y + 3xy$ at the point $(1,1)$ as you move in the specified directions:

① $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

Notice $\hat{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ is already a unit vector since $\|\hat{u}\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$

So we calculate $\vec{\nabla} f = \langle f_x, f_y \rangle = \langle 2xy + 3y, x^2 + 3x \rangle$ and at $(1,1)$ we have $\vec{\nabla} f(1,1) = \langle 2(1)(1) + 3(1), (1)^2 + 3(1) \rangle = \langle 2+3, 1+3 \rangle = \langle 5, 4 \rangle$.

Then $D_{\hat{u}} f = \vec{\nabla} f \cdot \hat{u} = \langle 5, 4 \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{5}{\sqrt{2}} + \frac{4}{\sqrt{2}} = \boxed{\frac{9}{\sqrt{2}}}$

② $\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$

Again, this is a unit vector already: $\sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = 1$.

So, $D_{\hat{u}} f = \langle 5, 4 \rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = 3 + \frac{16}{5} = \boxed{\frac{31}{5}}$.

③ $\langle 4, 1 \rangle$

This is <u>not</u> a unit vector. If $\vec{u} = \langle 4, 1 \rangle$, then $\|\vec{u}\| = \sqrt{4^2 + 1^2} = \sqrt{17} \neq 1$.

So $\hat{u} = \frac{\vec{u}}{\|\vec{u}\|} = \left\langle \frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}} \right\rangle$ is the unit vector we want. Then,

$$D_{\hat{u}} f = \vec{\nabla} f \cdot \hat{u} = \langle 5, 4 \rangle \cdot \left\langle \frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}} \right\rangle = \frac{20}{\sqrt{17}} + \frac{4}{\sqrt{17}} = \boxed{\frac{24}{\sqrt{17}}}.$$

12 Recall that we've used the idea of a tangent plane or approximation to define $\vec{\nabla} f$. The directional derivative gives another interpretation. For one, note that $D_{\hat{u}} f = \vec{\nabla} f \cdot \hat{u} = \|\vec{\nabla} f\| \cdot \|\hat{u}\| \cos\theta = \|\vec{\nabla} f\| \cos\theta$ since $\|\hat{u}\| = 1$. So, at a specific point, $D_{\hat{u}} f$ is maximized if $\theta = 0$, i.e. if $\hat{u}$ and $\vec{\nabla} f$ point in the same direction. In that case, we also get $D_{\hat{u}} f = \|\vec{\nabla} f\| \cos(0) = \|\vec{\nabla} f\|$, so $\|\vec{\nabla} f\|$ is the rate of change in that direction.

By identical reasoning, we see that the minimum rate of change is $-\|\vec{\nabla} f\|$ and happens when $\hat{u}$ and $\vec{\nabla} f$ point in <u>opposite</u> directions. Finally, if $\theta = 90°$ then $\vec{\nabla} f \cdot \hat{u}$ is zero. That is, if you move perpendicular to $\vec{\nabla} f$, $D_{\hat{u}} f = 0$, so the function value <u>does not change</u>.

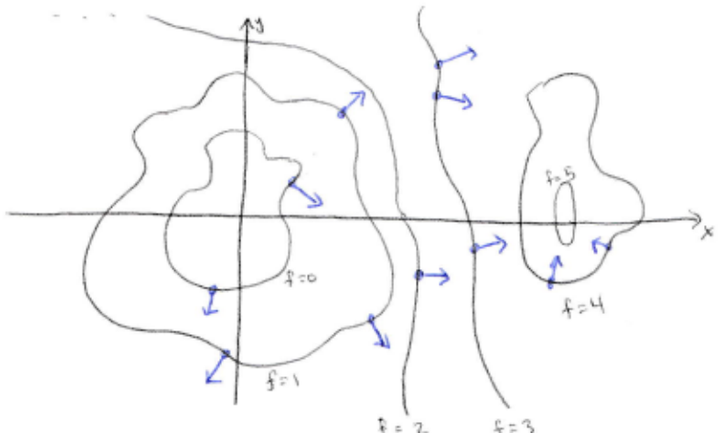
In other words, if we consider a level curve $f(x, y) = K$, then $\vec{\nabla} f$ should be perpendicular to it.

SUMMARY:

- $\vec{\nabla} f$ points in the direction of greatest increase of the function f,
- $\|\vec{\nabla} f\|$ is the rate of increase in that direction, and
- $\vec{\nabla} f$ is perpendicular to level sets $f = K$.

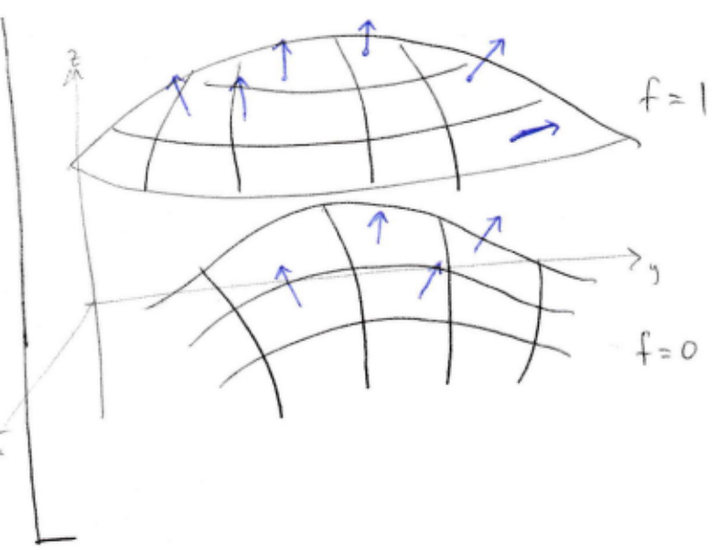
13 This can help us "see" the gradient in some circumstances

Examples: Draw $\vec{\nabla} f$ at a few points, given the level sets drawn below:

① 

[Solutions in blue for clarity]

Note $\vec{\nabla} f$ (at each point) is perpendicular to the given level curve, and points in the direction of <u>increase</u>.



Note again that $\vec{\nabla} f$ is perpendicular (now to level surfaces $f(x,y,z) = k$) and points in the direction of increase.