§12.6

Main Concepts

1. The Chain Rule
2. Mnemonic
3. An Important Case

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1 Let's consider an example. Say $f(x,y) = x^2\cos(x+y)$ measures the temperature at a location (x,y). If I'm moving through space, maybe I'm not so concerned with the temperature everywhere, but just my own temperature at time t. So, I describe my location as $x(t) = e^t$ and $y(t) = t^2 - 1$ at time t.

Then, my temperature at time t is:

$$f(x(t), y(t)) = (e^t)^2 \cos\left((e^t) + (t^2-1)\right).$$

This is a mess, and taking derivatives can be challenging. On the other hand, it's useful to know (how quickly am I heating up?). The (multivariable) chain rule helps to find this derivative:

CHAIN RULE (this case) $$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Notice the similarity to the usual chain rule, except we have partial derivatives and multiple terms.

Example: For the above setup, find $\frac{df}{dt}$ at time $t=0$.

At $t=0$, we have $x = e^0 = 1$ and $y = (0)^2 - 1 = -1$.

Then

$$\frac{\partial f}{\partial x} = 2x\cos(x+y) - x^2\sin(x+y) \xrightarrow{@(x,y)=(1,-1)} \frac{\partial f}{\partial x} = 2(1)\cos(1-1) - (1)^2\sin(1-1)$$
$$= 2\cos(0) - \sin(0) = 2$$

$$\frac{\partial f}{\partial y} = -x^2\sin(x+y) \xrightarrow{@(x,y)=(1,-1)} \frac{\partial f}{\partial y} = -(1)^2\sin(1-1) = -\sin(0) = 0$$

$$\frac{dx}{dt} = e^t \xrightarrow{@\,t=0} \frac{dx}{dt} = e^0 = 1$$

$$\frac{dy}{dt} = 2t \xrightarrow{@\,t=0} \frac{dy}{dt} = 2(0) = 0$$

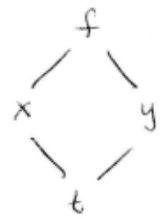
So, all together:

$$\left.\frac{df}{dt}\right|_{t=0} = (2)(1) + (0)(0) = \boxed{2}$$

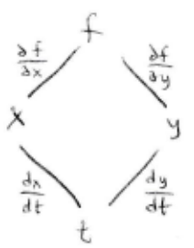
If we have a function $f(x,y,z)$ and x,y,z are themselves given by functions $x(t)$, $y(t)$, $z(t)$, we also get a chain rule:

CHAIN RULE (3-variable to 1-variable case) $\quad \dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial f}{\partial z}\cdot\dfrac{dz}{dt}$.

More generally, we can write down a chain rule wherever we have a function depending on variables that are themselves functions (and so on). A useful mnemonic device is to create dependence graphs like the one drawn here. This is telling us f is a function of x & y, and they are each functions of t.



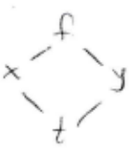
To get the chain rule, label each edge with the derivative between them, then follow every path from top to bottom. Each path gives a term, the product of the edge labels. This will also work for more "layers" and more variables, as we'll now illustrate:



Examples: For the given functions, write a formula for the requested derivative (but don't compute it).

(1) $f(x,y) = x^2 y$, $x(t) = \cos t$, $y(t) = t+1$; $\dfrac{df}{dt}$.

This is just like our example above, but let's draw the dependence graph:

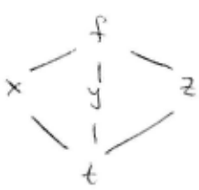


So, $\dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\cdot\dfrac{dy}{dt}$

"left path" "right path"

(2) $f(x,y,z) = x+yz$, $x(t) = t+1$, $y(t) = t^2$, $z(t) = t^3$; $\dfrac{df}{dt}$

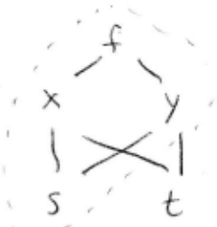
Same idea:



, so $\dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial f}{\partial z}\dfrac{dz}{dt}$.

(3) $f(x,y) = xy$, $x(s,t) = s+t$, $y(s,t) = s^2+t+1$; $\dfrac{\partial f}{\partial s}$

Similar idea, but we have more than one "base" variable now. But since we want $\dfrac{\partial f}{\partial s}$, we only look at this portion of the diagram.



So, we get

$$\boxed{\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}}$$

④ $f(x,y) = xy$, $x(t) = t^2$, $y(t) = \cos(t)$, $t(u,v) = u+v^2$; $\frac{\partial f}{\partial u}$

This example gives a "taller" graph:



But again the idea is the same:

$$\boxed{\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{dx}{dt}\frac{\partial t}{\partial u} + \frac{\partial f}{\partial y}\frac{dy}{dt}\frac{\partial t}{\partial u}}$$

And, for concreteness, let's solve at least one of those:

Example: For ③ above, compute $\frac{\partial f}{\partial s}$ in terms of s and t.

We have

$$\frac{\partial f}{\partial x} = y \qquad \frac{\partial x}{\partial s} = 1$$

$$\frac{\partial f}{\partial y} = x \qquad \frac{\partial y}{\partial s} = 2s$$

So $$\frac{\partial f}{\partial s} = (y)(1) + (x)(2s) = y + 2sx$$

To put it in terms of s,t, use the formulae for x,y:

$$\boxed{\frac{\partial f}{\partial s} = (s^2+t+1) + 2s(s+t) = 3s^2 + t + 1 + 2st}$$

3 Finally, there's a specific case (in fact the ones we did above) that's useful to remember and sometimes write differently. When we have $f(x,y)$, $x(t)$, $y(t)$, then we found $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$. Writing $\vec{r}(t) = \langle x(t), y(t)\rangle$ like before, this can now be written as a dot product:

$$\frac{df}{dt} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle = \vec{\nabla} f \cdot \vec{r}\,'(t)$$

and the same works if we have a z-coordinate as well.

This looks (perhaps) more like the chain rule we're used to:

$$\left(f(\vec{r}(t))\right)' = \vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}\,'(t).$$

Example: Let's do the same example as the beginning with this notation.

I.e. $f(x,y) = x^2\cos(x+y)$, $x(t) = e^t$, $y(t) = t^2 - 1$ @ $t = 0$.

Now $\vec{r}(t) = \langle e^t, t^2 - 1 \rangle$, ~~and so~~ $\vec{\nabla} f = \langle 2x\cos(x+y) - x^2\sin(x+y), -x^2\sin(x+y) \rangle$

and $\vec{r}\,'(t) = \langle e^t, 2t \rangle$, so

$$\left(f(\vec{r}(t))\right)'\Big|_{t=0} = \vec{\nabla} f(\vec{r}(0)) \cdot \vec{r}\,'(0)$$

$$= \vec{\nabla} f\left(e^0, (0)^2 - 1\right) \cdot \langle e^0, 2(0) \rangle$$

$$= \vec{\nabla} f(1, -1) \cdot \langle 1, 0 \rangle$$

$$= \langle 2(1)\cos(1-1) - (1)^2\sin(1-1),\ -(1)^2\sin(1-1) \rangle \cdot \langle 1, 0 \rangle$$

$$= \langle 2, 0 \rangle \cdot \langle 1, 0 \rangle$$

$$= (2)(1) + (0)(0) = \boxed{2}.$$