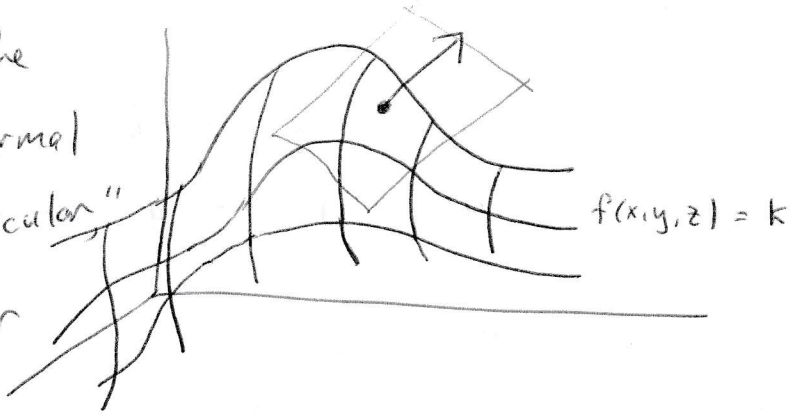
§12.7

Main Concept

I Tangent Planes, again again

I We've noted that the gradient is perpendicular to level sets. This is useful for finding a tangent plane to a surface. Indeed, one way we've described surfaces is a level set of some function $f(x,y,z)$, say $f(x,y,z) = k$. Then, at a point (a,b,c) on the surface, to describe the tangent plane we need a normal vector. But "normal = perpendicular", so $\vec{\nabla} f$ is a normal vector we can use.



Examples: Find an equation for the tangent plane to the described surfaces at the given point.

① $x^2 + y^2 + z^2 = 9$ at $(2,-2,1)$

Let $f(x,y,z) = x^2 + y^2 + z^2$. Then our surface is the level set $f = 9$.

So, the gradient is

$$\vec{\nabla} f = \langle 2x, 2y, 2z \rangle$$

and the normal is

$$\vec{\nabla} f(2,-2,1) = \langle 2(2), 2(-2), 2(1) \rangle = \langle 4, -4, 2 \rangle.$$

So the plane has equation $4x - 4y + 2z = D$ and plugging in $(2,-2,1)$ gives $D = 4(2) - 4(-2) + 2(1) = 8 + 8 + 2 = 18$.

$$\boxed{4x - 4y + 2z = 18}$$

(2) $x - y + z = 1$ at $(1,1,1)$

This is already a plane, so it should be its own tangent plane. We can check this:

$f(x,y,z) = x - y + z$

$\vec{\nabla} f = \langle 1, -1, 1 \rangle$ $\rightsquigarrow$ $\vec{N} = \vec{\nabla} f(1,1,1) = \langle 1, -1, 1 \rangle$ so the plane is

$x - y + z = D$ and $(1,1,1)$ gives $D = 1 - 1 + 1 = 1$ so $\boxed{x - y + z = 1}$

(3) $z = 4xy^3$ at $(2,1,8)$

Notice this isn't given as a level surface. But rewriting as $z - 4xy^3 = 0$ makes it a level surface for $f(x,y,z) = z - 4xy^3$ at $f = 0$. Then

$\vec{\nabla} f = \langle -4y^3, -12xy^2, 1 \rangle$ so $\vec{N} = \vec{\nabla} f(2,1,8) = \langle -4(1)^3, -12(2)(1)^2, 1 \rangle$

$= \langle -4, -24, 1 \rangle$.

So the plane is $-4x - 24y + z = D$ and using $(2,1,8)$:

$D = -4(2) - 24(1) + (8) = -24$ i.e. $\boxed{-4x - 24y + z = -24}$

(4) [Partial example from §12.4]

$z = x^2 y$ at $(1,2,2)$

Same idea: let $f(x,y,z) = x^2 y - z$. Then this is $f = 0$, so

$\vec{\nabla} f = \langle 2xy, x^2, -1 \rangle$ and $\vec{N} = \vec{\nabla} f(1,2,2) = \langle 2(1)(2), (1)^2, -1 \rangle = \langle 4, 1, -1 \rangle$

So $4x + y - z = D$ and $D = 4(1) + (2) - (2) = 4$

So $\boxed{4x + y - z = 4}$