# §12.8

## Main Concepts

1. Local Extrema
2. 2nd Derivative Test
3. Boundaries

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1 In the single-variable case, being able to compute derivatives had a crucial application: finding maxima and minima. We do the same here. Our first result is the same as in the single-variable case:

**THEOREM** If $f(x,y)$ [or $f(x,y,z)$] is an extremum [minimum or maximum], then one of the following must be true:

- $\vec{\nabla} f(x,y) = \vec{0}$
- $\vec{\nabla} f(x,y)$ doesn't exist
- $(x,y)$ is a boundary point.

We usually have differentiable functions, so $\vec{\nabla} f$ exists. Otherwise the remaining cases are the usual ones $\vec{\nabla} f = \vec{0}$ (derivative = 0) or we are at the boundary ("endpoints").

Example: Find all (local) extrema of $f(x,y) = 10 - x^4 - 2y^2 + y$

We don't have a boundary here, so we set the derivative equal to zero:

$$\vec{0} = \vec{\nabla} f := \langle -4x^3, -4y + 1 \rangle$$

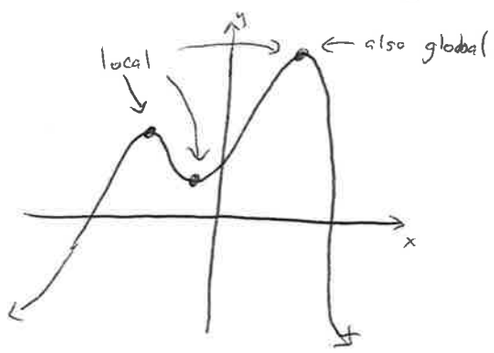
so $-4x^3 = 0 \longrightarrow x = 0$

$-4y + 1 = 0 \longrightarrow y = \frac{1}{4}$.

So we only have the one local extremum at $(0, \frac{1}{4})$.

Note: this example asks for "local" extrema since we're not sure if this is the maximum/minimum on the whole surface.

This is again in analogy with the one-variable case.



This now suggests two follow-up questions: how can we tell if a local extremum is a max or a min, and how do we find global extrema? This is the goal of the rest of §12.8.

2 In the 1-variable case, we distinguished local maxima from local minima using the second derivative. We do the same here, but we have more 2nd derivatives to consider.

**2nd Derivative Test** Suppose $\vec{\nabla} f(x,y) = \vec{0}$ for some $(x,y)$. Define

$$D = \begin{vmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{vmatrix} \quad \text{(this determinant)}$$

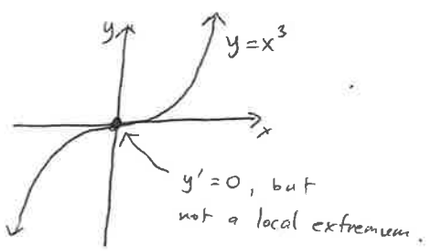
(I) If $D>0$ and $f_{xx}(x,y)>0$, then $(x,y)$ is a local minimum.

(II) If $D>0$ and $f_{xx}(x,y)<0$, then $(x,y)$ is a local maximum.

(III) If $D<0$, then $(x,y)$ is a saddle point.

(IV) If $D=0$, no conclusion can be reached.

Notice that case (III) suggests that $\vec{\nabla} f = 0$ doesn't automatically mean we have a local extremum. In the single-variable case, this is akin to having an inflection point.




Examples: Find and classify the local extrema for each of the following functions.

① $f(x,y) = x^2+2x+y^2-8y$

As before, start with: $\vec{0} = \vec{\nabla} f = \langle 2x+2, 2y-8 \rangle$, so $2x+2=0 \to x=-1$, $2y-8=0 \to y=4$

giving the one point $(-1,4)$. Now, write the matrix $D$ above:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$$

and $f_{xx} = 2 > 0$. So $(-1,4)$ is a local minimum.

(2) $f(x,y) = xy^2 + 10x^2 + 5y^2$

Again, solve: $\vec{0} = \vec{\nabla} f = \langle y^2 + 20x, 2xy + 10y \rangle$. The second component is $0 = 2xy + 10y = (2x+10)y$ so we have two cases.

I) $y = 0$. Then the first component gives $0 = y^2 + 20x = (0)^2 + 20x = 20x$ So $x = 0$. $\underline{(0,0)}$

II) $y \neq 0$: Then we must have $2x + 10 = 0$ so $x = -5$. Plugging into the first component again gives: $0 = y^2 + 20x = y^2 + 20(-5) = y^2 - 100$. So $y^2 = 100$ and $y = \pm 10$: $\underline{(-5,10)}$ $\underline{(-5,-10)}$

Now we need second derivatives. Write

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 20 & 2y \\ 2y & 10 \end{vmatrix}$$ Consider each point:

$(0,0)$: $D = \begin{vmatrix} 20 & 0 \\ 0 & 10 \end{vmatrix} = 200 > 0$ and $20 > 0$ $\rightarrow$ local min

$(-5,10)$: $D = \begin{vmatrix} 20 & 20 \\ 20 & 10 \end{vmatrix} = 200 - 400 = -200 < 0$ $\rightarrow$ saddle point

$(-5,-10)$: $D = \begin{vmatrix} 20 & -20 \\ -20 & 10 \end{vmatrix} = 200 - 400 = -200 < 0$ $\rightarrow$ saddle point

3 Finally, we discuss global extrema. In general, there's no reason to expect a maximum or minimum to exist: in 1D, the function $f(x) = x$ grows to $\pm\infty$ in each direction. However, we have the following fact:

**EXTREME VALUE THEOREM** If $f$ is continuous on a closed and bounded domain, then $f$ has a global max & a global min.

While this is nice, it adds more complication for us, since a bounded domain will have a boundary, and we've already seen that extrema can occur along the boundary (without $\vec{\nabla} f = \vec{0}$).

Examples: Find the global extrema of $f$ on the given domain

(1) $f(x,y) = x^2 + y^2$ on $R = \{(x,y) \mid -1 \leq x \leq 2, -1 \leq y \leq 3\}$.

First, let's look for $\vec{\nabla} f = \vec{0}$:

$$\vec{0} = \vec{\nabla} f = \langle 2x, 2y \rangle \longrightarrow \begin{matrix} 2x = 0 \\ 2y = 0 \end{matrix} \longrightarrow \begin{matrix} x = 0 \\ y = 0 \end{matrix} \qquad \underline{(0,0)}.$$

Second, let's look at the boundary. There are 4 "pieces" of the boundary

I) $x = -1$, $-1 \le y \le 3$

$f(-1, y) = (-1)^2 + y^2 = y^2 + 1$. To find critical points, set the derivative $= 0$ and consider the boundary (of this boundary).

derivative $\longrightarrow$ $2y = 0 \longrightarrow y = 0$ $\qquad \underline{(-1, 0)}$

boundary $\longrightarrow$ $y = -1, \; y = 3$ $\qquad \underline{(-1, -1)}$ $\qquad \underline{(-1, 3)}$

II) $x = 2$ $\quad -1 \le y \le 3$ $\quad$ Same idea: $f(2, y) = y^2 + 4$

$d/dy \longrightarrow$ $2y = 0 \longrightarrow y = 0$ $\qquad \underline{(2, 0)}$

boundary $\longrightarrow$ $y = -1, \; y = 3$ $\qquad \underline{(2, -1)}$ $\qquad \underline{(2, 3)}$

III) $y = -1$ $\quad -1 \le x \le 2$ $\quad$ Same: $f(x, -1) = x^2 + 1$

$d/dx \longrightarrow$ $2x = 0$ $\qquad \underline{(0, -1)}$

boundary $\longrightarrow$ $x = -1, \; x = 2$ [already got these above]

IV) $y = 3$ $\quad -1 \le x \le 2$ $\quad f(x, 3) = x^2 + 9$

$d/dx \longrightarrow$ $2x = 0 \longrightarrow x = 0$ $\qquad \underline{(0, 3)}$

boundary $\longrightarrow$ $x = -1, \; x = 2$ [already got these, too].

Finally, we have our list of 9 possible extrema points. To finish, we check each one:

$f(0,0) = 0^2 + 0^2 = 0$ $\qquad f(-1,0) = (-1)^2 + 0^2 = 1$ $\qquad f(-1,-1) = (-1)^2 + (-1)^2 = 2$

$f(-1,3) = (-1)^2 + 3^2 = 10$ $\qquad f(2,0) = 2^2 + 0^2 = 4$ $\qquad f(2,-1) = 2^2 + (-1)^2 = 5$

$f(2,3) = 2^2 + 3^2 = 13$ $\qquad f(0,-1) = 0^2 + (-1)^2 = 1$ $\qquad f(0,3) = 0^2 + 3^2 = 9$

So, the minimum is 0, occurring at $(0,0)$, and the maximum is 13, occurring at $(2,3)$

(Note: the minimum has $\vec{\nabla} f = 0$, but the maximum is along the boundary)

(2) $f(x,y) = x^2 - 2y^2$ on $S = \{(x,y) \mid x^2 + y^2 \le 4\}$.

Same approach. Start with $\vec{\nabla} f = \vec{0}$:

$\vec{0} = \vec{\nabla} f = \langle 2x, -4y \rangle \longrightarrow x = y = 0 \longrightarrow \underline{(0,0)}$.

Now consider the boundary, which only has one "piece": $x^2 + y^2 = 4$.

Then $y^2 = 4 - x^2$, so

$f(x,y) = x^2 - 2(4-x^2) = 3x^2 - 8$. Since $x^2 = 4 - y^2$ is between 0 and 4, we have $-2 \le x \le 2$. So we consider the 1-variable derivative and the boundary of the boundary:

$\xrightarrow{d/dx}$ $6x = 0 \longrightarrow x = 0 \longrightarrow y^2 = 4 - x^2 = 4 - 0^2 = 4 \longrightarrow y = \pm 2$

$(0,2)$ $(0,-2)$.

$\xrightarrow{\text{boundary}}$ $x = \pm 2 \longrightarrow y^2 = 4 - (\pm 2)^2 = 4 - 4 = 0$

$(2,0)$ $(-2,0)$

Lastly, we test all of our (5) points:

$f(0,0) = (0)^2 - 2(0)^2 = 0$

$f(0,2) = (0)^2 - 2(2)^2 = -8$

$f(0,-2) = (0)^2 - 2(-2)^2 = -8$

$f(2,0) = (2)^2 - 2(0)^2 = 4$

$f(-2,0) = (-2)^2 - 2(0)^2 = 4$

So, the minimum is $-8$ at the points $(0,2)$ and $(0,-2)$, and the maximum is $4$ at the points $(2,0)$ and $(-2,0)$.