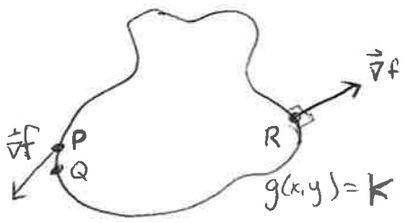
§12.9

Main Concept

1 Lagrange multipliers

1 We saw in the previous section that the problem of optimizing a function on a closed & bounded domain can be tedious when checking the boundary. The method of Lagrange multipliers can sometimes provide an easier alternative.

The idea is as follows: suppose we want to maximize $f(x,y)$ but we are constrained by an equation $g(x,y)=k$. The gradient $\vec{\nabla}f$ tells us which direction to "move" to increase f. But the level curve $g=k$ might stop us from moving that way:



In the drawn example, the point P isn't a maximum, because even though we can't move in the $\vec{\nabla}f$ direction without leaving the level curve, moving from P to Q should still give some increase, since $\vec{\nabla}f$ has some component in the $\vec{PQ}$ direction.

On the other hand, R might be a maximum; since $\vec{\nabla}f$ is perpendicular to the level curve, we can't increase f just by "sliding" left or right.

So, we're looking for points where $\vec{\nabla}f$ is perpendicular to $g(x,y)=k$. But we already know that $\vec{\nabla}g$ is perpendicular to $g(x,y)=k$ (see §12.5), So, we want $\vec{\nabla}f$ and $\vec{\nabla}g$ to point in the same direction. In other words, we should be able to scale $\vec{\nabla}g$ by some constant to get $\vec{\nabla}f$. I.e.:

LAGRANGE MULTIPLIERS Each critical point of f constrained by $g(\vec{p})=k$ satisfies $\vec{\nabla}f=\lambda\vec{\nabla}g$ for some λ.

Examples: Find all critical points of f subject to the given constraint.

① $f(x,y)=x^2-2y^2$ with $x^2+y^2=4$.

[Note: this is the same as the last example in §12.8].

As suggested, set up the system of equations $\vec{\nabla}f=\lambda\vec{\nabla}g$ for $g(x,y)=x^2+y^2$.

This gives $\langle 2x, -4y \rangle = \vec{\nabla} f = \lambda \vec{\nabla} g = \lambda \langle 2x, 2y \rangle$, so:

$$2x = 2\lambda x$$
$$-4y = 2\lambda y$$
$$x^2 + y^2 = 4$$

Notice I've restated the constraint, since this now gives 3 equations in 3 unknowns $(x, y, \& \lambda)$. Let's eliminate λ since we don't need it (we only want to know x and y). We can do this in the first equation by dividing by $2x$. However, we should worry about division by zero. So, we have 2 cases:

I) $x \neq 0$ Now we can divide without worrying. The first equation gives

$$\lambda = \frac{2x}{2x} = 1$$

Plugging into the second gives

$$-4y = 2(1)y = 2y \longrightarrow 0 = 6y \longrightarrow \boxed{y = 0}$$

And plugging this into the third gives

$$x^2 + (0)^2 = 4 \longrightarrow x = \pm 2.$$

So we've found $\boxed{(2,0)}$ and $\boxed{(-2,0)}$

II) $x = 0$ we can't solve the first equation for λ now, but actually things are even easier: plug into the third equation directly to get

$$(0)^2 + y^2 = 4 \longrightarrow y = \pm 2$$

I.e. we have the points $\boxed{(0,2)}$ and $\boxed{(0,-2)}$.

(2) $f(x,y,z) = x + 2y + 3z$ subject to $z = \sqrt{4 - x^2 - y^2}$.

Note that our constraint is not written as a level curve. One option to remedy this is to write $z - \sqrt{4 - x^2 - y^2} = 0$. Another is to square both sides first, and then rewrite: $z^2 = 4 - x^2 - y^2 \longrightarrow x^2 + y^2 + z^2 = 4$, remembering that $z \geq 0$. I'll use the second: $g(x,y,z) = x^2 + y^2 + z^2$.

Then:

$$\langle 1, 2, 3 \rangle = \vec{\nabla} f = \lambda \vec{\nabla} g = \lambda \langle 2x, 2y, 2z \rangle. \quad \text{So}$$

$z \geq 0$, $\quad 1 = 2\lambda x$, $\quad 2 = 2\lambda y$, $\quad 3 = 2\lambda z$, and $x^2 + y^2 + z^2 = 4$

We can't have $\lambda = 0$, since the first 3 equations would be $1 = 0, 2 = 0, 3 = 0$.

So, we can divide by λ to solve for x, y, z:

$$x = \frac{1}{2\lambda} \qquad y = \frac{2}{2\lambda} \qquad z = \frac{3}{2\lambda}.$$

Plugging into the 4th equation gives:

$$\frac{1}{(2\lambda)^2} + \frac{4}{(2\lambda)^2} + \frac{9}{(2\lambda)^2} = 4$$

$$14 = 16\lambda^2$$

$$\lambda = \pm\frac{\sqrt{14}}{4}$$

Since $z = \frac{3}{2\lambda} \geq 0$, we must have $\lambda > 0$ as well, so $\lambda = \frac{\sqrt{14}}{4}$.

Then

$$x = \frac{1}{2\lambda} = \frac{2}{\sqrt{14}}$$

$$y = \frac{2}{2\lambda} = \frac{4}{\sqrt{14}}$$

$$z = \frac{3}{2\lambda} = \frac{6}{\sqrt{14}}$$

is our only critical point.