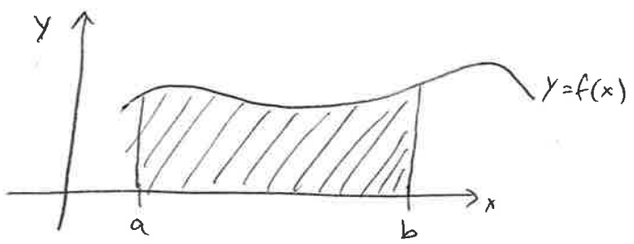
§13.1

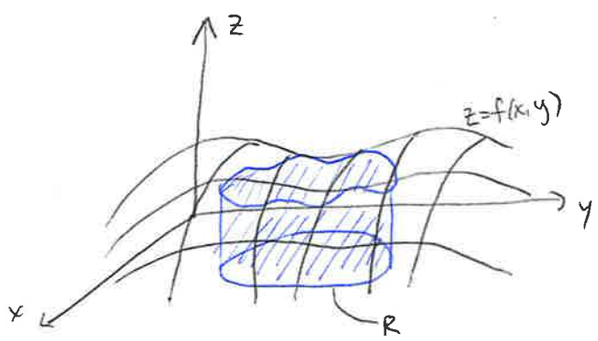
Main Concept

1 Double integration

1 Recall that in the single-variable setting, the integral $\int_a^b f(x)\,dx$ computed the



area under $y=f(x)$ between a and b. We'd like to generalize this. If $f(x,y)$ is a function of two variables, then $z=f(x,y)$ is a surface, and

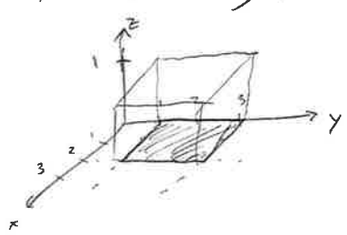


for any region R in the domain (i.e. lying in the xy-plane), we can ask how much volume is above R and below $z=f(x,y)$. We will denote this volume by $\iint_R f(x,y)\,dA$, the <u>double-integral of f on R</u>.

Examples: Compute $\iint_R f(x,y)\,dA$ for each of the following:

① $f(x,y)=1$ $\quad R=\{(x,y) \mid 0\le x\le 2,\ 1\le y\le 3\}$.

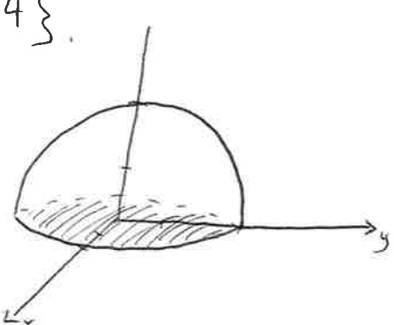
Let's sketch a picture:



The region R is shaded, and since $z=f(x,y)=1$, the height is constant. So, the volume is just the volume of this box: $\iint_R f(x,y)\,dA = (2)(2)(1) = 4$.

② $f(x,y)=\sqrt{4-x^2-y^2}$ $\quad R=\{(x,y) \mid x^2+y^2\le 4\}$.

Again, we sketch:



Since $z=\sqrt{4-x^2-y^2}$,

$\rightarrow z^2 = 4-x^2-y^2$

$\rightarrow x^2+y^2+z^2=4$ is a sphere of radius 2.

since $z = +\sqrt{4-x^2-y^2}$, we only get the positive (top-half) of the sphere, so the volume is half that of a sphere: $\iint_R f(x,y)\,dA = \frac{1}{2}\left[\frac{4}{3}\pi(2)^3\right] = \frac{16}{3}\pi$.

Notice that doubling the height doubles the volume, and stacking two objects on top of each other adds their volume. So, we get the usual rules for integrals:

$$\iint_R c\,f(x,y)\,dA = c\iint_R f(x,y)\,dA \qquad \text{[pull out constants]}$$

$$\iint_R \left[f(x,y)+g(x,y)\right]dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA \qquad \text{[split up sums]}$$

Example: Compute $\iint_R 7\sqrt{4-x^2-y^2}\,dA$ for $R = \{(x,y) \mid x^2+y^2 \le 4\}$.

We have:

$$\iint_R 7\sqrt{4-x^2-y^2}\,dA = 7\iint \sqrt{4-x^2-y^2}\,dA = 7\left(\frac{16}{3}\pi\right) = \frac{112}{3}\pi$$

(where here we used the previous example)