§ 13.2

Main Concepts

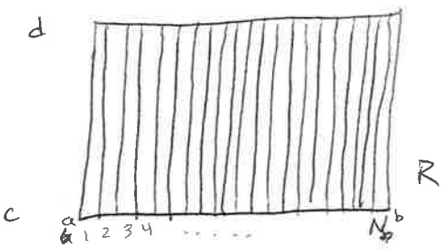
1. Iterated integrals
2. Order of integration
3. Separable integrals

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1 When a picture isn't enough, we'll need another way to compute double integrals.

To start, we'll do the case when R is a rectangle: $R = \{(x,y) \mid \begin{matrix} a \le x \le b \\ c \le y \le d \end{matrix}\}$.

To find the volume $V = \iint_R f(x,y)\,dA$, let's first cut R into thin strips of width Δx.



The total volume is obtained by adding up all of these smaller volumes. To find each smaller one, we can integrate y from c to d to get a cross-sectional area, and multiply by Δx. So, we expect $V \approx \sum_{i=1}^{N} \left(\int_c^d f(x_i,y)\,dy\right)\Delta x$. Taking the limit as $\Delta x \to 0$ gives $V = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$. We usually omit the brackets:

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

Examples: Compute the following iterated integrals.

① $\int_2^4 \int_0^3 (x^2+y)\,dy\,dx$

The first (inside) integral is with respect to y, so x is a constant:

$$\int_2^4 \int_0^3 (x^2+y)\,dy\,dx = \int_2^4 \left(x^2 y + \frac{y^2}{2}\right)\Big|_{y=0}^{3} dx$$

$$= \int_2^4 \left(x^2(3-0) + \tfrac{1}{2}(3^2-0^2)\right) dx \qquad \text{[Plug in the } y\text{-bounds]}$$

$$= \int_2^4 \left(3x^2 + \frac{9}{2}\right) dx$$

$$= 3\frac{x^3}{3} + \frac{9}{2}x \Big|_2^4 = (4^3-2^3) + \frac{9}{2}(4-2) = 64 - 8 + 9$$

$$= 65$$

(2) $\int_0^3 \int_1^2 xy\,dy\,dx$

Same idea: integrate one at a time:

$$\int_0^3 \int_1^2 xy\,dy\,dx = \int_0^3 \left(x\,\frac{y^2}{2}\right)\Big|_{y=1}^{2} dx = \int_0^3 \frac{x}{2}(2^2-1^2)\,dx .$$

$$= \frac{3}{2}\int_0^3 x\,dx = \frac{3}{2}\,\frac{x^2}{2}\Big|_{x=0}^{3} = \frac{3}{4}(3^2-0^2) = \frac{27}{4}$$

(3) $\int_0^{\pi/2} \int_0^3 x\cos(xy)\,dy\,dx$

Notice we need a u-substitution for the y-integral. This is fine; continue to treat x like a constant. Let $u = xy$ so $du = x\,dy$, giving

$$\int_0^{\pi/2} \int_0^3 x\cos(xy)\,dy\,dx = \int_0^{\pi/2} \int_{y=0}^{y=3} \cos(u)\,du\,dx = \int_0^{\pi/2} \sin(u)\Big|_{y=0}^{3} dx =$$

$$= \int_0^{\pi/2} \sin(xy)\Big|_{y=0}^{3} dx = \int_0^{\pi/2} (\sin(3x) - \sin(0))\,dx =$$

$$= \int_0^{\pi/2} \sin(3x)\,dx = -\frac{1}{3}\cos(3x)\Big|_{x=0}^{\pi/2} = -\frac{1}{3}\left(\cos\left(\frac{3\pi}{2}\right) - \cos(0)\right) =$$

$$= -\frac{1}{3}(0 - (+1)) = \frac{1}{3}$$

[2] In our original argument, we cut up R into vertical strips, but we could just as easily have done horizontal strips. This would give:

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy .$$

Sometimes, one order of integration is easier than the other.

Examples: Do all of the above with the reversed order of integration.

(1) $\int_2^4 \int_0^3 (x^2+y)\,dy\,dx = \int_0^3 \int_2^4 (x^2+y)\,dx\,dy = \int_0^3 \left(\frac{x^3}{3} + xy\right)\Big|_{x=2}^{4} dy = \int_0^3 \left[\frac{1}{3}(4^3-2^3) + y(4-2)\right] dy$

$$= \int_0^3 \left(\frac{56}{3} + 2y\right) dy = \frac{56}{3}y + \frac{2y^2}{2}\Big|_{y=0}^{3} = \frac{56}{3}(3-0) + (3^2-0^2)$$

$$= 56 + 9 = 65$$

(2) $\int_0^3 \int_1^2 xy\,dy\,dx = \int_1^2 \int_0^3 xy\,dx\,dy = \int_1^2 \frac{x^2}{2}y\Big|_{x=0}^{3} dy = \int_1^2 \frac{9}{2}y\,dy =$

$$= \frac{9}{4} y^2 \Big|_{y=1}^{2} = \frac{9}{4}\left(2^2 - 1^2\right) = \frac{27}{4}$$

(3) $$\int_0^{\pi/2} \int_0^3 x\cos(xy)\, dy\, dx = \int_0^3 \int_0^{\pi/2} x\cos(xy)\, dx\, dy.$$

$$= \int_0^3 \left[\frac{x}{y}\sin(xy)\Big|_{x=0}^{\pi/2} - \int_0^{\pi/2} \frac{1}{y}\sin(xy)\, dx \right] dy$$

(integration by parts: $u = x$, $dv = \cos(xy)\,dx$)

$$= \int_0^3 \left(\frac{\pi}{2y}\sin\left(\frac{\pi}{2}y\right) + \frac{1}{y^2}\cos(xy)\Big|_{x=0}^{\pi/2} \right) dy$$

$$= \int_0^3 \left[\frac{\pi}{2y}\sin\left(\frac{\pi}{2}y\right) + \frac{1}{y^2}\left(\cos\left(\frac{\pi}{2}y\right) - 1\right) \right] dy$$

$$= ?$$

This integral is hard enough that I'll stop here: this highlights how useful it can be to pick the "right" order of integration.

[3] Finally, I'll note a useful technique. Some integrands can be written as a product of "x" terms and "y" terms separately. I.e. we have integrals of the form $\int_a^b \int_c^d f(x)g(y)\, dy\, dx$. We can simplify these:

$$\int_a^b \int_c^d f(x)g(y)\, dy\, dx = \int_a^b \left(f(x) \int_c^d g(y)\, dy \right) dx$$

[$f(x)$ is a constant for the dy integral]

$$= \left(\int_c^d g(y)\, dy \right) \int_a^b f(x)\, dx$$

[$\int_c^d g(y)\,dy$ is a constant for the dx integral]

$$= \left(\int_a^b f(x)\, dx \right) \left(\int_c^d g(y)\, dy \right).$$

Example: Use this technique on example (2) above.

$$\int_0^3 \int_1^2 xy\, dy\, dx = \left(\int_0^3 x\, dx \right) \left(\int_1^2 y\, dy \right)$$

$$= \left(\frac{x^2}{2}\Big|_{x=0}^{3} \right) \left(\frac{y^2}{2}\Big|_{y=1}^{2} \right)$$

$$= \frac{1}{4}\left(3^2 - 0^2\right)\left(2^2 - 1^2\right) = \frac{1}{4}(9)(3) = \frac{27}{4}$$