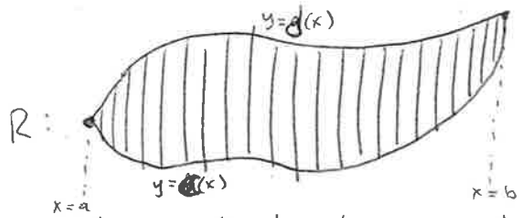
§13.3

Main Concepts

1. Nonrectangular regions
2. Order of integration
3. Splitting regions

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1. With rectangles out of the way, we move to more general regions. As before, we

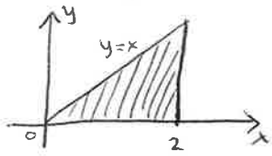


cut our region into vertical stripes. Now, the inner integral (with respect to dy) has bounds that depend on x. I.e. our integral is of the form

$$\iint_R f(x,y)\,dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx.$$

Examples: Compute $\iint_R f(x,y)\,dA$ for each of the following.

① $f(x,y) = x+y$ R is this shaded region:



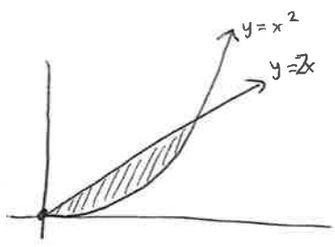
In the x-direction, we start at 0 and end at 2.
Notice that the vertical stripe corresponding to "x" starts at $y=0$ and ends at $y=x$. So, the integral is

$$\iint_R f(x,y)\,dA = \int_0^2 \int_0^x (x+y)\,dy\,dx$$

$$= \int_0^2 \left(xy + \frac{y^2}{2}\right)\Big|_{y=0}^{x} dx = \int_0^2 \left[x(x-0) + \frac{1}{2}(x^2-0^2)\right] dx$$

$$= \int_0^2 \left(x^2 + \frac{1}{2}x^2\right) dx = \int_0^2 \frac{3}{2}x^2\,dx = \frac{3}{2}\,\frac{x^3}{3}\Big|_{x=0}^{2} = \frac{1}{2}\left(2^3 - 0^3\right) = 4.$$

② $f(x,y) = xy$ R is this shaded region:



This time, we don't have the x-bounds. But we can find them: we're looking for the intersection of $y=2x$ and $y=x^2$.
So $2x = x^2$ and we get $x=0$ or $x=2$. So the two intersections are (0,0) and (2,4). So x goes from 0 to 2 and y goes from x^2 to $2x$:

$$\iint_R f(x,y)\,dA = \int_0^2 \int_{x^2}^{2x} xy\,dy\,dx = \int_0^2 x\,\frac{y^2}{2}\Big|_{y=x^2}^{2x} dx = \int_0^2 \frac{x}{2}\left((2x)^2 - (x^2)^2\right) dx$$

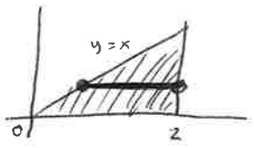
$= \frac{1}{2}\int_0^2 (4x^3 - x^5)\,dx = \frac{1}{2}\left(\frac{4x^4}{4} - \frac{x^6}{6}\right)\Big|_{x=0}^{2} = \frac{1}{2}(2^4 - 0^4) - \frac{1}{12}(2^6 - 0^6) = 8 - \frac{16}{3}$

$= \frac{8}{3}$

2 As before, we can switch the order of integration, potentially simplifying things.

Examples: Do the previous examples with $\int dy$ on the outside.

① Note that our picture doesn't give the upper bound for y. But we can find it: $y = x$ and $x = 2$, so $y = 2$. So, y goes from 0 to 2. For a fixed y, x starts at $x = y$ and stops at $x = 2$ (see the dark band above). So:



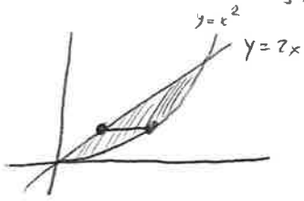
$$\iint_R f(x,y)\,dA = \int_0^2 \int_y^2 (x+y)\,dx\,dy = \int_0^2 \left(\frac{x^2}{2} + xy\right)\Big|_{x=y}^{2} dy$$

$$= \int_0^2 \left(\frac{1}{2}(2^2 - y^2) + y(2-y)\right) dy = \int_0^2 \left(2 - \frac{1}{2}y^2 + 2y - y^2\right) dy$$

$$= \int_0^2 \left(-\frac{3}{2}y^2 + 2y + 2\right) dy = \left(-\frac{1}{2}y^3 + y^2 + 2y\right)\Big|_{y=0}^{2}$$

$$= -\frac{1}{2}(2)^3 + (2)^2 + 2(2) = -4 + 4 + 4 = 4$$

② We already solved the y-bounds last time: $y = 0$ to $y = 4$. For a fixed y, we start at $y = 2x \longrightarrow x = \frac{1}{2}y$ and end at $y = x^2 \longrightarrow x = \sqrt{y}$. So:



$$\iint_R f(x,y)\,dA = \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} xy\,dx\,dy = \int_0^4 \left(\frac{x^2}{2}y\right)\Big|_{x=\frac{1}{2}y}^{\sqrt{y}} dy = \int_0^4 \frac{1}{2}y\left((\sqrt{y})^2 - \left(\frac{1}{2}y\right)^2\right) dy$$

$$= \int_0^4 \frac{1}{2}y\left(y - \frac{1}{4}y^2\right) dy = \frac{1}{2}\int_0^4 \left(y^2 - \frac{1}{4}y^3\right) dy = \frac{1}{6}y^3 - \frac{1}{32}y^4\Big|_{y=0}^{4}$$

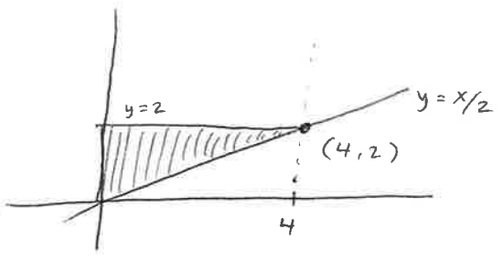
$$= \frac{1}{6}4^3 - \frac{1}{32}4^4 = \frac{32}{3} - 8 = \frac{8}{3}$$

Notice that we got the same answers as previously.

Sometimes, switching the order of integration is necessary to solve an integral:

Example: Find $\int_0^4 \int_{x/2}^{2} e^{y^2}\,dy\,dx$.

Notice that the usual strategy fails here since e^{y^2} doesn't have a nice antiderivative. We'll try switching the order of integration, so let's first sketch the region: $0 \le x \le 4$ and $x/2 \le y \le 2$:



Then we can switch the order:

$$\int_0^4 \int_{x/2}^{2} e^{y^2}\, dy\, dx = \int_0^2 \int_0^{2y} e^{y^2}\, dx\, dy$$

$$= \int_0^2 x e^{y^2} \Big|_{x=0}^{2y} dy = \int_0^2 e^{y^2}(2y-0)\, dy = \int_0^2 2y e^{y^2}\, dy$$

(Let $u = y^2$, $du = 2y\, dy$)

$$= \int_0^4 e^u du = e^u \Big|_{u=0}^{4} = e^4 - 1.$$

Let's practice:

In-class practice: Reverse the order of integration by drawing the region.

① $\int_0^1 \int_0^{4x} (x+2y)\, dy\, dx$

② $\int_{-3}^{3} \int_{|x|}^{3} xy\, dy\, dx$

③ $\int_0^1 \int_0^{\sqrt{1-y^2}} 5\, dx\, dy$

④ $\int_{-2}^{2} \int_0^{\sqrt{4-x^2}} 6\, dy\, dx$

Answers:

① $\int_0^4 \int_{y/4}^{1} (x+2y)\, dx\, dy$

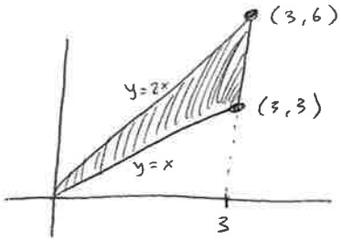
② $\int_0^3 \int_{-y}^{y} xy\, dx\, dy$

③ $\int_0^1 \int_0^{\sqrt{1-x^2}} 5\, dy\, dx$

④ $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} 6\, dx\, dy$

3 Finally, we should note that not every region can be written in the way we've been doing.

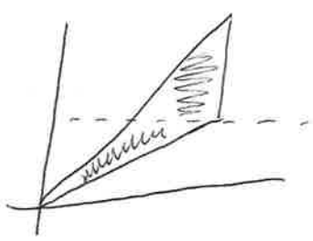
Example: Set up $\iint_R f(x,y)\,dA$ using iterated integrals in 2 ways for the region R shaded here:



If we do this as we did before with vertical stripes, nothing changes: x goes from 0 to 3, and for a fixed x, y goes from x to 2x:

$$\iint_R f(x,y)\,dA = \int_0^3 \int_x^{2x} f(x,y)\,dy\,dx.$$

If we try the other way, then y goes from 0 to 6, but x is more complicated: for part of R, x goes from $y/2$ to y, and for the rest of R, x goes from $y/2$ to $x=3$. In order to handle the "switch", we break up R into two regions:



Then:

$$\iint_R f(x,y)\,dA$$

$$= \underbrace{\int_0^3 \int_{y/2}^{y} f(x,y)\,dx\,dy}_{1^{st}\text{ region}} + \underbrace{\int_3^6 \int_{y/2}^{3} f(x,y)\,dx\,dy}_{2^{nd}\text{ region}}.$$

Further, some regions have to be split up no matter which way we integrate first, e.g.:

