§13.4

Main concept

1. Polar coordinates
2. Converting between coordinates.

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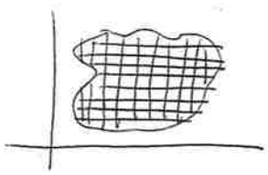
1. For some regions, integrating with respect to x & y is awkward because the region is better described in polar by r and θ. E.g. if R is a disk of radius 3 centered at $(0,0)$, then the x,y bounds would be:

$$-3 \le x \le 3 \quad \text{and} \quad -\sqrt{9-x^2} \le y \le \sqrt{9-x^2}.$$

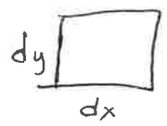
While the polar bounds would be

$$0 \le r \le 3 \quad \text{and} \quad 0 \le \theta \le 2\pi.$$

The only additional thing we need to understand is dA. When we cut R into x & y pieces, we get:

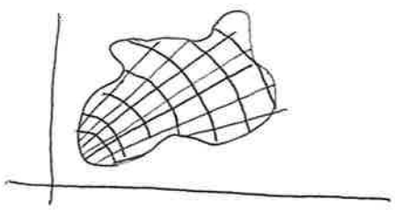


so each small piece looks like:



and has area $dA = dx \cdot dy$.

For polar, we cut along r = constant (circles) and θ = constant (rays) to get:



so each small piece looks like



For $d\theta$ and dr small, this is roughly a rectangle with side lengths dr and $r \cdot d\theta$. So, we get $\boxed{dA = r \cdot dr \cdot d\theta}$.

Example: Derive the volume of a cylinder of radius 3 and height 5 by integrating in polar coordinates.

This is the same as computing $\iint_R 5\,dA$ where R is a disk of radius 3. So:

$$V = \iint_R 5\,dA = \int_0^{2\pi}\int_0^3 5\,r\,dr\,d\theta = 5\left(\int_0^{2\pi} d\theta\right)\left(\int_0^3 r\,dr\right) = 5(\theta)\Big|_{\theta=0}^{2\pi}\left(\frac{r^2}{2}\right)\Big|_{r=0}^{3}$$

$$= \frac{5}{2}(2\pi - 0)(3^2 - 0^2) = 5\pi \cdot 3^2 = 45\pi.$$

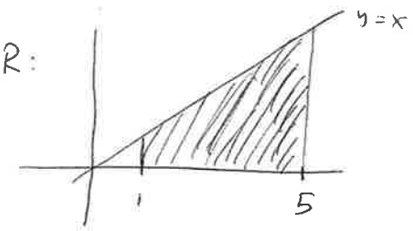
Notice how much harder this would be in x,y coordinates.

2 It is often useful to switch between coordinates. This is nothing more than an application of §11.9.

Examples: Solve by switching coordinate systems.

① $\int_1^5 \int_0^x \frac{3}{x^2+y^2}\, dy\, dx$

Note this is R:



In other words $0 \le \theta \le \frac{\pi}{4}$, and $x = r\cos\theta$, so

$1 \le x \le 5 \longrightarrow \frac{1}{\cos\theta} \le r \le \frac{5}{\cos\theta}$. So:

$$\int_1^5 \int_0^x \frac{3}{x^2+y^2}\, dy\, dx = \int_0^{\pi/4} \int_{\sec\theta}^{5\sec\theta} \frac{3}{r^2}\, r\, dr\, d\theta = \int_0^{\pi/4} 3\log|r| \Big|_{r=\sec\theta}^{r=5\sec\theta} d\theta$$

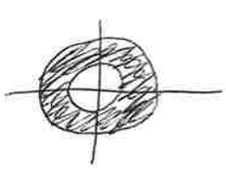
$$= \int_0^{\pi/4} 3\left(\log(5\sec\theta) - \log(\sec\theta)\right) d\theta$$

$$= \int_0^{\pi/4} 3\log\left(\frac{5\sec\theta}{\sec\theta}\right) d\theta$$

$$= \int_0^{\pi/4} 3\log(5)\, d\theta = 3\log(5)\,\theta \Big|_0^{\pi/4} = 3\log(5)\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{3}{4}\pi\log(5).$$

② $\iint_R e^{x^2+y^2}\, dA$ inside $R = \{(x,y) \mid 1 \le x^2+y^2 \le 4\}$



This is a disk of radius 2; minus a disk of radius 1.

So:

$$\iint_R e^{x^2+y^2}\, dA = \int_0^{2\pi} \int_1^2 e^{r^2}\, r\, dr\, d\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_1^2 e^{r^2} r\, dr\right)$$

Let $u = r^2$, $du = 2r\,dr$

$$= \left(\theta\Big|_0^{2\pi}\right)\left(\int_1^4 e^u \cdot \frac{1}{2}\, du\right) = \frac{1}{2}(2\pi - 0)\left(e^u\Big|_1^4\right)$$

$$= \pi\left(e^4 - e\right).$$