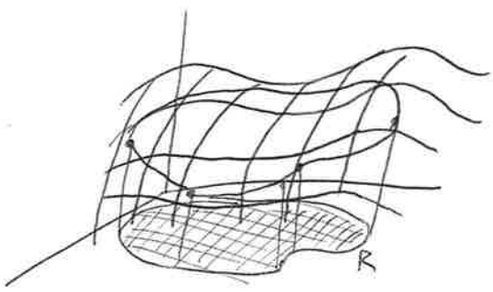
§13.6

Main Concept

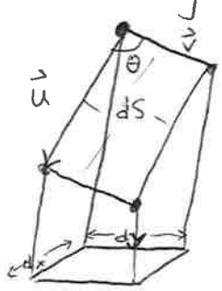
1 Surface area

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1 If $z = f(x,y)$ defines a surface, then we can ask whether its surface area can be computed. The double integral provides this tool. First, as usual, we cut up our surface into



pieces and look at each piece. Zooming into one of the small pieces of R gives:



We want to relate the area dS to the quantity $dA = dx\,dy$. Now that we've zoomed in, the area dS is the area of a parallelogram, so $dS = \|\vec{u}\|\,\|\vec{v}\|\sin\theta = \|\vec{u}\times\vec{v}\|$. So it would be useful to know the components of $\vec{u}$ and $\vec{v}$. For $\vec{u}$, we've moved dx units in the x-direction and 0 units in the y-direction. In the $z=f(x,y)$ direction, we should have slope $f_x = \frac{\partial f}{\partial x}$. So $f_x = \frac{\text{rise}}{\text{run}} = \frac{\text{rise}}{dx} \longrightarrow \text{rise} = f_x\,dx$. In other words,

$$\vec{u} = \langle dx, 0, f_x\,dx\rangle \qquad \text{and similarly} \qquad \vec{v} = \langle 0, dy, f_y\,dy\rangle.$$

So,

$$dS = \|\vec{u}\times\vec{v}\| = \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ dx & 0 & f_x\,dx \\ 0 & dy & f_y\,dy \end{vmatrix} \right\| = \|\langle -f_x\,dx\,dy, -f_y\,dx\,dy, dx\,dy\rangle\|$$

$$= dx\,dy\,\|\langle -f_x, -f_y, 1\rangle\| = \sqrt{(-f_x)^2 + (-f_y)^2 + 1^2}\,dx\,dy = \sqrt{f_x^2 + f_y^2 + 1}\,dA.$$

Integrating both sides gives our result:

$$\boxed{S = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\,dA}$$

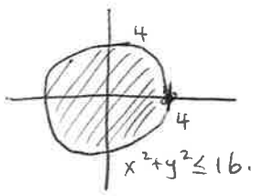
Examples: Find the surface areas described.

① The part of $z = x^2+y^2$ below $z=16$.



This is a paraboloid with $z = f(x,y) = x^2+y^2$.

Being below $z=16$ means we are interested in $x^2+y^2 \le 16$ for our bounds of integration. This is a disk, so we'll probably use polar. Then,



$f_x = 2x+0 = 2x$, $f_y = 0+2y = 2y$, so

$$S = \iint_R \sqrt{(2x)^2+(2y)^2+1}\,dA = \int_0^{2\pi}\int_0^4 \sqrt{4r^2+1}\; r\,dr\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^4 \sqrt{4r^2+1}\; r\,dr\right)$$

Let $u = 4r^2+1$

$du = 8r\,dr$

$r=0 \longrightarrow u = 4(0)^2+1 = 1$

$r=4 \longrightarrow u = 4(4)^2+1 = 65$

$$= \left(\theta\Big|_0^{2\pi}\right)\left(\int_1^{65} u^{1/2}\,\frac{du}{8}\right)$$

$$= \frac{2\pi-0}{8}\int_1^{65} u^{1/2}\,du = \frac{\pi}{4}\cdot\frac{u^{3/2}}{3/2}\Big|_1^{65} = \frac{\pi}{6}\left(65^{3/2}-1\right).$$

② The part of the plane $x+z=10$ above the disk $x^2+y^2 \le 1$.



Looking above $x^2+y^2 \le 1$ cuts out of a cylinder, and the angle of the plane makes an ellipse as shown.

We don't have $z=f(x,y)$, so solve for z: $z = 10-x$.

So, let $f(x,y) = 10-x$ giving $f_x = -1$ and $f_y = 0$. Then

$$S = \iint_R \sqrt{(-1)^2+0^2+1}\,dA = \int_0^{2\pi}\int_0^1 \sqrt{2}\; r\,dr\,d\theta$$

$$= \sqrt{2}\left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 r\,dr\right) = \sqrt{2}\,(2\pi)\left(\tfrac{1}{2}r^2\Big|_0^1\right) = \pi\sqrt{2}\left(1^2-0^2\right) = \pi\sqrt{2}.$$