§13.7

Main Concepts

1. Triple integrals
2. Regions of integration
3. Center of mass

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1. With double integrals established, we continue ~~[illegible]~~ generalizing in this direction. Given a function $f(x,y,z)$ and a volume in (3D-)space, we can compute $\iiint_R f(x,y,z)\,dV$ analogously as we did in 2D.

First, a few notes on interpretation. Just as $\iint_R dA$ gave us the area of a region R in the xy-plane, $\iiint_R dV$ gives the volume of a region R in xyz-space.

The other case with a clear physical interpretation is when $f(x,y,z)$ is a density of some kind. For example, if R is an object (of non-uniform density) and $f(x,y,z)$ measures the density (say, in kg/m^3) then $\iiint_R f(x,y,z)\,dV$ gives the total mass. The same is true if f gives charge density (C/m^3), population density ($organisms/m^3$), or any other density ($units/m^3$).

Evaluating such an integral is the same as before: one at a time.

Examples: Evaluate the following iterated integrals.

(1) $\int_{-1}^{1}\int_{0}^{1}\int_{1}^{3} 2xy\,dx\,dy\,dz$

We'll integrate from the inside out:

$$\int_{-1}^{1}\int_{0}^{1}\int_{1}^{3} 2xy\,dx\,dy\,dz = \int_{-1}^{1}\int_{0}^{1} x^2y\Big|_{x=1}^{3} dy\,dz = \int_{-1}^{1}\int_{0}^{1} (3^2-1^2)y\,dy\,dz = \int_{-1}^{1}\int_{0}^{1} 8y\,dy\,dz =$$

$$= \int_{-1}^{1} 4y^2\Big|_{y=0}^{1} dz = \int_{-1}^{1} 4(1^2-0^2)\,dz = \int_{-1}^{1} 4\,dz = 4z\Big|_{z=-1}^{1} = 4(1-(-1)) = 8.$$

(2) $\int_{0}^{1}\int_{0}^{2}\int_{0}^{3-x-y} x\,dz\,dy\,dx$

We do the same. Notice that our bounds on z can depend on both x & y now.

$$\int_{0}^{1}\int_{0}^{2}\int_{0}^{3-x-y} x\,dz\,dy\,dx = \int_{0}^{1}\int_{0}^{2} xz\Big|_{z=0}^{3-x-y} dy\,dx = \int_{0}^{1}\int_{0}^{2} x(3-x-y-0)\,dy\,dx =$$

$= \int_0^1 \int_0^2 (3x - x^2 - xy)\, dy\, dx = \int_0^1 (3xy - x^2 y - \frac{1}{2}xy^2)\Big|_{y=0}^{2} dx = \int_0^1 [3x(2-0) - x^2(2-0) - \frac{1}{2}x(2^2-0^2)]\, dx =$

$= \int_0^1 (6x - 2x^2 - 2x)\, dx = \int_0^1 (4x - 2x^2)\, dx = 2x^2 - \frac{2}{3}x^3\Big|_{x=0}^{1} = 2(1^2-0^2) - \frac{2}{3}(1^3-0^3) =$

$= 2 - \frac{2}{3} = \frac{4}{3}$.

③ $\int_0^1 \int_0^{1-x} \int_0^{x^2+y^2} dz\, dy\, dx$

Notice now we have variables in multiple bounds. The innermost integral can have bounds depending on any of the other variables, while further out integrals can't depend on the inner ones. Computationally, things are the same:

$\int_0^1 \int_0^{1-x} \int_0^{x^2+y^2} dz\, dy\, dx = \int_0^1 \int_0^{1-x} z\Big|_{z=0}^{x^2+y^2} dy\, dx = \int_0^1 \int_0^{1-x} (x^2+y^2)\, dy\, dx =$

$= \int_0^1 \left(x^2 y + \frac{1}{3}y^3\right)\Big|_{y=0}^{1-x} dx = \int_0^1 \left[x^2(1-x) + \frac{1}{3}(1-x)^3\right] dx =$

$= \int_0^1 (x^2 - x^3)\, dx + \frac{1}{3}\int_0^1 (1-x)^3\, dx =$

Let $u = 1-x$, $du = -dx$, $x=0 \longrightarrow u=1$, $x=1 \longrightarrow u=0$ in the second integral

$= \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_{x=0}^{1} + \frac{1}{3}\int_1^0 u^3(-du) =$

$= \frac{1}{3} - \frac{1}{4} + \frac{1}{3}\int_0^1 u^3\, du = \frac{1}{3} - \frac{1}{4} + \frac{1}{3}\cdot\frac{u^4}{4}\Big|_{u=0}^{1} = \frac{4}{12} - \frac{3}{12} + \frac{1}{12} = \frac{1}{6}$.

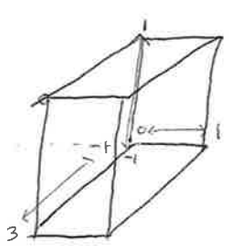
12 Iterated triple integrals are supposed to describe a region in space; how can we convert back-and-forth between a description of R and an iterated integral to calculate $\iiint_R$? The idea is the same as before: we cut up the region and describe the resulting bounds.

Examples: Describe the regions of integration in each of the previous examples above.

① $\int_{-1}^{1}\int_0^1\int_1^3$ ___ $dx\, dy\, dz$ (ignoring the integrand).

This has $-1 \le z \le 1$, $0 \le y \le 1$, and $1 \le x \le 3$

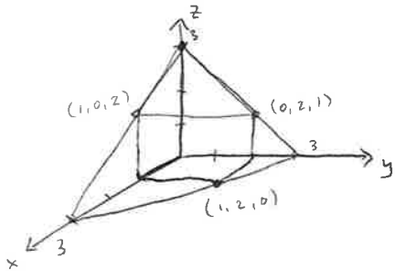
so it's a rectangular prism.



② $\int_0^1 \int_0^2 \int_0^{3-x-y}$ ___ $dz\, dy\, dx$.

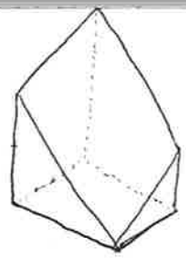
Here $0 \le x \le 1$, $0 \le y \le 2$, and $0 \le z \le 3-x-y$.

- x and y describe a rectangle in the xy-plane.
- $z=0$ is the xy-plane, and



- $z = 3-x-y$ can be written as

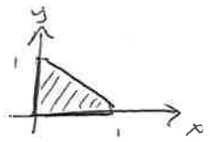
$x+y+z=3$, so this is another plane (tilted).



So overall, the region looks like:

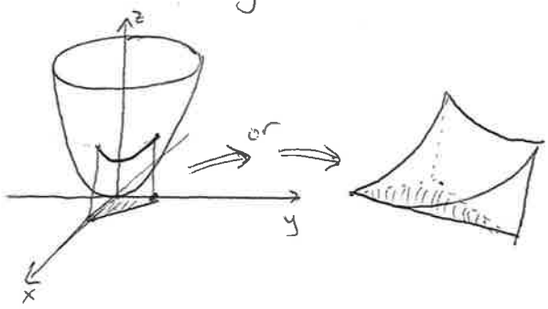
③ $\int_0^1 \int_0^{1-x} \int_0^{x^2+y^2} \text{—} \, dz\, dy\, dx$

we have $0 \le x \le 1$, $0 \le y \le 1-x$, and $0 \le z \le x^2+y^2$. Let's start with the xy-plane. Here we have $y = 1-x$ as a bound,



which is a line of slope -1. So we get this shaded region. For each point in this triangle, we let z go from 0 (the xy-plane) to $z = x^2+y^2$ (a paraboloid).

So, overall, the region looks like the ~~part of~~ space between this triangle and the paraboloid, i.e.



We can also try to go the otherway:

Examples: Set up a triple integral to find the volume of R for each case below.

① R is the region below $z = x^2+y^2$, above the xy-plane, ~~[illegible]~~ within the first octant, and bounded by $x+y \le 10$.

Converting these, we get $0 \le x \le 10$, $0 \le y \le 10-x$, and $0 \le z \le x^2+y^2$.

So $V = \int_0^{10} \int_0^{10-x} \int_0^{x^2+y^2} dz\, dy\, dx$.

Alternatively, we could write $0 \le y \le 10$, $0 \le x \le 10-y$, and $0 \le z \le x^2+y^2$, giving:

$$V = \int_0^{10} \int_0^{10-y} \int_0^{x^2+y^2} dz\, dx\, dy,$$

and there are other expressions for V with the other orders of integration.

② R is a solid sphere of radius 1 centered at the origin.

At most, we'll have $-1 \le x \le 1$. If we recall that the sphere has equation $x^2+y^2+z^2=1$, then we can also find bounds for y. Clearly $-1 \le y \le 1$, but

not all those y-values work for every x. In fact, we must have $x^2+y^2 \le 1$ since $z^2 \geq 0$,

so $y^2 \le 1-x^2$ and $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$. Similarly, for a given x & y,

$z^2 \le 1-x^2-y^2$, so $-\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}$. So, the volume of the sphere is

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\, dy\, dx.$$

(Side note: wouldn't it be nice to be able to do this in spherical coordinates?)

~~③ R is a pyramid with vertices (1,1,0), (0,1,1), (1,0,1) and (1,1,0).~~

3 Finally, let's end with a physical example. Suppose $f(x,y,z)$ measures the density of an object at a point (x,y,z). Then, as noted, $M = \iiint_R f(x,y,z)\,dV$ is the total mass.

Also define $M_x = \iiint_R x f(x,y,z)\,dV$, $M_y = \iiint_R y f(x,y,z)\,dV$, and $M_z = \iiint_R z f(x,y,z)\,dV$.

Then the object's <u>center of mass</u> has coordinates $(\bar{x}, \bar{y}, \bar{z})$ given by

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_x}{M}, \frac{M_y}{M}, \frac{M_z}{M}\right).$$

Example: Find the center of mass if the density is given by $f(x,y,z) = z$ of an object bounded by $0 \le x \le 1$, $0 \le y \le 2$, and $0 \le z \le x^2+y^2$.

First, we find the mass:

$$M = \int_0^1 \int_0^2 \int_0^{x^2+y^2} z\, dz\, dy\, dx = \frac{1}{2}\int_0^1 \int_0^2 z^2 \Big|_{z=0}^{x^2+y^2} dy\, dx = \frac{1}{2}\int_0^1 \int_0^2 \left[(x^2+y^2)^2 - 0\right] dy\, dx =$$

$$= \frac{1}{2}\int_0^1 \int_0^2 (x^4 + 2x^2y^2 + y^4)\, dy\, dx = \frac{1}{2}\int_0^1 \left(x^4 y + \frac{2}{3}x^2y^3 + \frac{1}{5}y^5 \Big|_{y=0}^{2}\right) dx = \frac{1}{2}\int_0^1 \left(2x^4 + \frac{16}{3}x^2 + \frac{32}{5}\right) dx =$$

$$= \int_0^1 \left(x^4 + \frac{8}{3}x^2 + \frac{16}{5}\right) dx = \frac{x^5}{5} + \frac{8}{9}x^3 + \frac{16}{5}x \Big|_{x=0}^{1} = \frac{1}{5} + \frac{8}{9} + \frac{16}{5} = \frac{193}{45}$$

Then we find the moments (M_x, M_y, M_z):

$$M_x = \int_0^1 \int_0^2 \int_0^{x^2+y^2} x z\, dz\, dy\, dx = \int_0^1 x \int_0^2 \frac{1}{2} z^2 \Big|_{z=0}^{x^2+y^2} dy\, dx = \frac{1}{2}\int_0^1 x \int_0^2 (x^4 + 2x^2y^2 + y^4)\, dy\, dx =$$

$$= \frac{1}{2}\int_0^1 x\left(x^4 y + \frac{2}{3}x^2y^3 + \frac{1}{5}y^5\right)\Big|_{y=0}^{2} dx = \frac{1}{2}\int_0^1 \left(2x^5 + \frac{16}{3}x^3 + \frac{32}{5}x\right) dx = \frac{x^6}{6} + \frac{8}{3}\cdot\frac{x^4}{4} + \frac{16}{5}\frac{x^2}{2}\Big|_{x=0}^{1}$$

$$= \frac{1}{6} + \frac{2}{3} + \frac{8}{5} = \frac{73}{30}$$

So $\bar{x} = \frac{M_x}{M} = \frac{73}{30} \cdot \frac{45}{193} = \boxed{\frac{219}{386}}$.

$$M_y = \int_0^1 \int_0^2 \int_0^{x^2+y^2} y z\, dz\, dy\, dx = \frac{1}{2}\int_0^1 \int_0^2 y(x^4 + 2x^2y^2 + y^4)\, dy\, dx = \frac{1}{2}\int_0^1 \int_0^2 (x^4 y + 2x^2y^3 + y^5)\, dy\, dx =$$

$$= \frac{1}{2}\int_0^1 \left(\frac{1}{2}x^4y^2 + \frac{2}{4}x^2y^4 + \frac{1}{6}y^6\right)\Big|_{y=0}^{2} dx = \frac{1}{2}\int_0^1 \left(2x^4 + 8x^2 + \frac{32}{3}\right) dx =$$

$$= \int_0^1 \left(x^4 + 4x^2 + \frac{16}{3}\right) dx = \frac{1}{5} + \frac{4}{3} + \frac{16}{3} = \frac{103}{15}$$

so $\bar{y} = \frac{103}{15} \cdot \frac{45}{193} = \boxed{\frac{309}{193}}$.

$$M_z = \int_0^1 \int_0^2 \int_0^{x^2+y^2} z^2\, dz\, dy\, dx = \frac{1}{3}\int_0^1 \int_0^2 (x^2+y^2)^3\, dy\, dx = \frac{1}{3}\int_0^1 \int_0^2 \left(x^6 + 3x^4y^2 + 3x^2y^4 + y^6\right) dy\, dx =$$

$$= \frac{1}{3}\int_0^1 \left(x^6 y + x^4 y^3 + \frac{3}{5}x^2y^5 + \frac{1}{7}y^7\right)\Big|_{y=0}^{2} dx = \frac{1}{3}\int_0^1 \left(2x^6 + 8x^4 + \frac{96}{5}x^2 + \frac{128}{7}\right) dx = \frac{1}{3}\left(\frac{2}{7} + \frac{8}{5} + \frac{32}{3} + \frac{128}{7}\right)$$

$$= \frac{1}{3} \cdot \frac{3238}{105} = \frac{3238}{315}$$

So $\bar{z} = \frac{3238}{315} \cdot \frac{45}{193} = \boxed{\frac{3238}{1351}}$