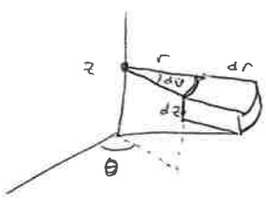
§13.8

Main Concepts

1. Cylindrical coordinates
2. Spherical coordinates.

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1 As with double-integrals, we can sometimes evaluate triple integrals more simply using a different coordinate system. In cylindrical coordinates, we divide our volume into chunks as we did in polar; only with 3 dimensions:



So, the small volume can be computed similarly: $dV = (dr)(dz)(r\,d\theta) = r\,dr\,d\theta\,dz$.

CYLINDRICAL INTEGRATION FACTOR $\quad dV = r\,dr\,d\theta\,dz$.

Example: Find the center of mass of the cylinder bounded by $x^2+y^2 \le 4$ and $0 \le z \le 6$, with density $f(x,y,z) = z$.

In cylindrical, our bounds are $0 \le r \le 2$, $0 \le \theta \le 2\pi$, and $0 \le z \le 6$. Then:

$$M = \iiint_R f(x,y,z)\,dV = \int_0^{2\pi}\int_0^{2}\int_0^{6} z\,r\,dz\,dr\,d\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{2} r\,dr\right)\left(\int_0^{6} z\,dz\right) =$$

$$= (2\pi)\left(\frac{2^2}{2}\right)\left(\frac{6^2}{2}\right) = (2\pi)(2)(18) = 72\pi$$

$$M_x = \iiint_R x\,f(x,y,z)\,dV = \int_0^{2\pi}\int_0^{2}\int_0^{6} (r\cos\theta)\cdot z\cdot r\,dz\,dr\,d\theta = \left(\int_0^{2\pi}\cos\theta\,d\theta\right)\left(\int_0^{2} r^2\,dr\right)\left(\int_0^{6} z\,dz\right)$$

$$= \left(\sin(2\pi) - \sin(0)\right)\left(\int_0^{2} r^2\,dr\right)\left(\int_0^{6} z\,dz\right) = 0 \qquad \text{so } \bar{x} = 0.$$

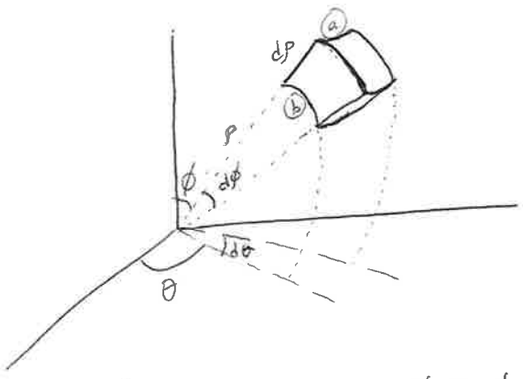
$$M_y = \int_0^{2\pi}\int_0^{2}\int_0^{6} (r\sin\theta)\cdot z\cdot r\,dz\,dr\,d\theta = 0 \quad \text{since } \int_0^{2\pi}\sin\theta\,d\theta = 0, \text{ too. So } \bar{y} = 0.$$

$$M_z = \int_0^{2\pi}\int_0^{2}\int_0^{6} z\cdot z\cdot r\,dz\,dr\,d\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{2} r\,dr\right)\left(\int_0^{6} z^2\,dz\right) = (2\pi)\left(\frac{2^2}{2}\right)\left(\frac{6^3}{3}\right) =$$

$$= (2\pi)(2)(72) = 288\pi. \qquad \text{So } \bar{z} = \frac{M_z}{M} = \frac{288\pi}{72\pi} = 4.$$

So the center of mass is $(0,0,4)$.

2 We can do the same for spherical coordinates. For this, we need to compute a small volume dV at a point (ρ,θ,ϕ) with a small change $d\rho$, $d\theta$, $d\phi$ in each coordinate.



We can approximate this as a rectangular prism, so the volume is the product of the three sides. One side has length $d\rho$, and the side labeled (b) has length $\rho\, d\phi$ since its an arc of a circle. Side (a) also is an arc of a circle, but it's a smaller circle centered on the z-axis with radius $\rho\sin\phi$, so it has length $\rho\sin\phi\, d\theta$.

Putting these all together gives:

SPHERICAL INTEGRATION FACTOR: $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$.

Example: Find the volume of a sphere of radius 1.

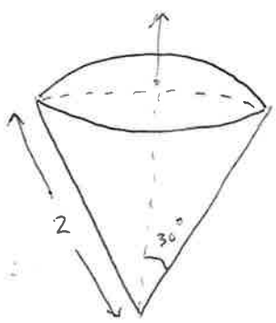
Note: we set up this integral in Cartesian (xyz) back in §13.7 and it was messy.

Here it's not so bad:

$$\text{Volume} = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin\phi\, d\phi\right)\left(\int_0^{1}\rho^2 d\rho\right) =$$

$$= \left(\theta\Big|_{0}^{2\pi}\right)\left(-\cos\phi\Big|_{\phi=0}^{\pi}\right)\left(\frac{1}{3}\rho^3\Big|_{\rho=0}^{1}\right) = (2\pi - 0)(\cos(0) - \cos(\pi))\left(\frac{1}{3} - 0\right) = \frac{4}{3}\pi.$$

Example: Find the volume of the spherical cone drawn:



Again $0 \le \theta \le 2\pi$ since we sweep through every angle around the z-axis. But $0 \le \phi \le 30^\circ = \frac{\pi}{6}$ since we get no further from the z-axis. Finally $0 \le \rho \le 2$, so:

$$\text{Volume} = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{2} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/6}\sin\phi\, d\phi\right)\left(\int_0^{2}\rho^2 d\rho\right) = (2\pi)\left(\cos(0) - \cos\left(\frac{\pi}{6}\right)\right)\left(\frac{1}{3}\cdot 2^3\right)$$

$$= (2\pi)\left(1 - \frac{\sqrt{3}}{2}\right)\left(\frac{8}{3}\right) = \frac{8}{3}\pi(2 - \sqrt{3})$$