§13.9

Main Concept

1. Integration factor (Jacobian)
2. Changing bounds
3. The Substitution Formula

1 When doing double- and triple-integrals, it has been useful at times to change to a more convenient coordinate system. For example, we've changed to polar, cylindrical, and spherical, and sometimes we've done the usual 1-dimensional u-substitution.

We'd like to generalize all of this. To do so, first recall the 1-d version. If we have a difficult integral $\int_a^b f(x)\,dx$, we'll sometimes make a substitution $x = x(u)$ for a new variable u. Then $dx = x'(u)\,du$ and our bounds $x=a$ & $x=b$ become $u = x^{-1}(a)$ and $u = x^{-1}(b)$. So, we get:

$$\int_{x^{-1}(a)}^{x^{-1}(b)} f(x(u))\, x'(u)\,du.$$

In particular, our integrand got multiplied by the integration factor $x'(u) = \dfrac{dx}{du}$.

Suppose now we have a double integral $\iint_R f(x,y)\,dx\,dy$. By analogy, if we write $x = x(u,v)$ and $y = y(u,v)$, then our bounds should change as above; $R \longrightarrow S$, and we should get an integration factor "$\dfrac{\partial(x,y)}{\partial(u,v)}$". This indeed happens, and the integration factor is usually called the Jacobian. We calculate it as follows:

JACOBIAN: If x & y are given in terms of u & v, the Jacobian is the determinant

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

For 3-variables it is defined similarly,

i.e. $$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Examples: Find the Jacobian in each of the following.

① $x = u+v$, $y = u-v$.

We start by computing partials: $\frac{\partial x}{\partial u} = 1$, $\frac{\partial x}{\partial v} = 1$, $\frac{\partial y}{\partial u} = 1$, $\frac{\partial y}{\partial v} = -1$. So

$$J = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (-1) - (1) = -2.$$

② $x = u^2 + v$, $y = u + v^2$

Do the same: $\frac{\partial x}{\partial u} = 2u$, $\frac{\partial x}{\partial v} = 1$, $\frac{\partial y}{\partial u} = 1$, $\frac{\partial y}{\partial v} = 2v$, So

$$J = \begin{vmatrix} 2u & 1 \\ 1 & 2v \end{vmatrix} = 4uv - 1.$$

③ $x = r\cos\theta$, $y = r\sin\theta$.

Notice this is the usual polar substitution and that we're therefore using r, θ instead of u, v for convenience. Then

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta, \text{ so}$$

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r.$$

(This is as expected, since we know $dx\,dy = dA = \underline{\underline{r}}\,dr\,d\theta$).

2 Next we need to understand how to change our bounds. This is probably best done through practice

Examples: Convert the xy-bounds to the new variables.

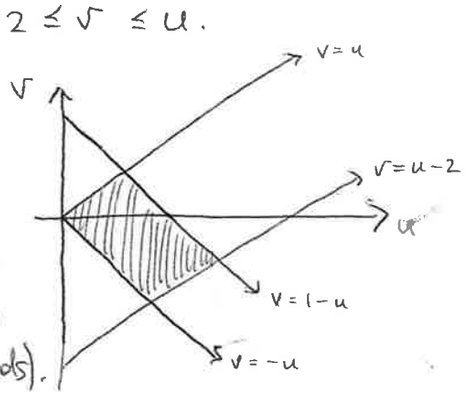
① $x = u+v$, $y = u-v$, $0 \le x \le 1$, $0 \le y \le 2$.

Directly plugging in gives $0 \le u+v \le 1$ and $0 \le u-v \le 2$.

This is a complete description. However, for integrating, it will be useful to have at least one variable bounded by constants. For this, note that adding $0 \le u+v$ and $0 \le u-v$ gives $0 \le 2u \longrightarrow u \ge 0$. Similarly, adding $u+v \le 1$ and $u-v \le 2$ gives $2u \le 3 \longrightarrow u \le \frac{3}{2}$. Solving the inequalities for v then gives $-u \le v \le 1-u$ and $u-2 \le v \le u$.

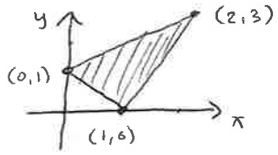
Overall, graphing is invaluable.

In the uv-plane, we get:



Note that this explains why the region is difficult to describe for integration (the slices in either direction have changing bounds).

(2) $x = \frac{u+v}{4}$, $y = \frac{3u-v}{4}$, (x,y) inside the triangle with vertices $(0,1)$, $(1,0)$, and $(2,3)$.



To start, let's write equations for each side of the triangle. For $(1,0)$ and $(0,1)$:

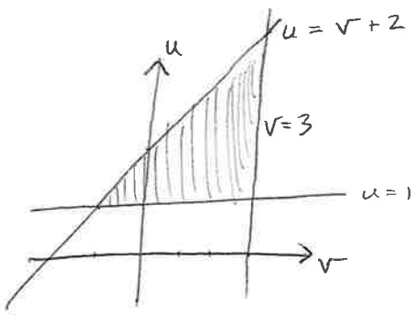
$y - 0 = \frac{0-1}{1-0}(x-1) = -(x-1) = 1-x$ and substituting gives

$$\frac{3u-v}{4} = 1 - \frac{u+v}{4} \iff 3u - v = 4 - u - v$$
$$\iff 4u = 4 \iff u = 1$$

This is one of our bounds. Next, for $(0,1)$ and $(2,3)$:

$y - 1 = \frac{3-1}{2-0}(x-0) = x$ so in u,v:

$$\frac{3u-v}{4} - 1 = \frac{u+v}{4} \iff 3u - v - 4 = u + v$$
$$\iff 2u - 2v = 4$$
$$\iff u = 2 + v$$



Finally, the last bound for $(1,0)$ and $(2,3)$ is:

$y - 0 = \frac{3-0}{2-1}(x-1) = 3(x-1) = 3x - 3$ so

$$\frac{3u-v}{4} = 3\frac{u+v}{4} - 3 \iff 3u - v = 3u + 3v - 12$$
$$\iff 12 = 4v$$
$$\iff v = 3.$$

We've sketched the new bounds. Notice they're much easier to integrate between now $\left(\int_{-1}^{3} \int_{1}^{v+2} - \, du \, dv \right)$.

(3) $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 \leq 1$.

This one is difficult algebraically, because we have lots of options, in fact! Thinking of this as the usual polar, we get $0 \leq r \leq 1$ and $0 \leq \theta < 2\pi$.

Algebraically, we get $r^2 \leq 1 \iff -1 \leq r \leq 1$ but no other conditions.

Here, it's useful to remember that we don't want redundancy - Since sin & cos are periodic, we only need $0 \leq \theta < 2\pi$. And if r is negative, we can add $180^\circ = \pi$ to get that point anyway $(-r\cos\theta = r\cos(\theta + \pi))$.

So we can assume $r \geq 0$. Notice $-1 \leq r \leq 1$, $0 \leq \theta < \pi$ would therefore also work.

3 Finally, let's put this all together into our (u,v)-substitution formula.

SUBSTITUTION FORMULA: Suppose x and y are given in terms of u, v, and that a region R in the xy-plane becomes S in the uv-plane. Then

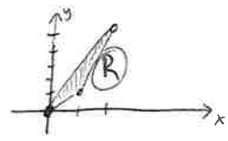
$$\iint_R f(x,y)\,dx\,dy = \iint_S f(u,v)\,|J|\,du\,dv$$

where J is the Jacobian.

Examples: Solve the given integrals by making a substitution.

① $\iint_R x\,dx\,dy$ where R is the triangle with vertices $(0,0)$, $(1,1)$, $(2,5)$, using $u = y - x$ and $v = 5x - 2y$.

In (x,y)-coordinates, the bounds are:



To convert the bounds, we'll need to write equations for each side. The bottom is the line $y = x$. The right has slope 4, and plugging in a point gives $y = 4x - 3$. Similarly, the left is $y = \frac{5}{2}x$. To rewrite them, solve for x, y in terms of u, v: $y = u + x \longrightarrow v = 5x - 2(u+x) = 5x - 2u - 2x = 3x - 2u$, so $3x = 2u + v$ and $\boxed{x = \frac{2}{3}u + \frac{1}{3}v}$. Plugging back in gives

$$\boxed{y = u + \left(\tfrac{2}{3}u + \tfrac{1}{3}v\right) = \tfrac{5}{3}u + \tfrac{1}{3}v.}$$

Now we plug this into each bound:

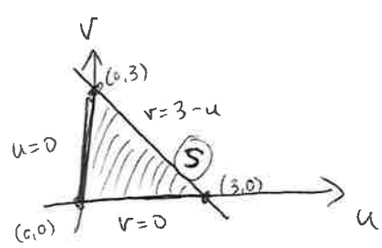
$y = x \longrightarrow \frac{5}{3}u + \frac{1}{3}v = \frac{2}{3}u + \frac{1}{3}v \longrightarrow \left(\frac{5}{3} - \frac{2}{3}\right)u = 0 \longrightarrow \underline{u = 0}.$

$y = 4x - 3 \longrightarrow \frac{5}{3}u + \frac{1}{3}v = 4\left(\frac{2}{3}u + \frac{1}{3}v\right) - 3 \longrightarrow \left(\frac{5}{3} - \frac{8}{3}\right)u = \left(\frac{4}{3} - \frac{1}{3}\right)v - 3$

$\longrightarrow -u = v - 3 \longrightarrow \underline{v = -u + 3}$

$y = \frac{5}{2}x \longrightarrow \frac{5}{3}u + \frac{1}{3}v = \frac{5}{2}\left(\frac{2}{3}u + \frac{1}{3}v\right) = \frac{5}{3}u + \frac{5}{6}v \longrightarrow \left(\frac{1}{3} - \frac{5}{6}\right)v = 0 \longrightarrow \underline{v = 0}.$

So, the (u,v) bounds are:



Lastly, to make the substitution, we need the Jacobian:

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2/3 & 1/3 \\ 5/3 & 1/3 \end{vmatrix} = \frac{2}{9} - \frac{5}{9} = -\frac{3}{9} = -\frac{1}{3}.$$

So, $$\iint_R x\,dx\,dy = \iint_S \left(\tfrac{2}{3}u + \tfrac{1}{3}v\right)|J|\,du\,dv = \int_0^3 \int_0^{3-u} \left(\tfrac{2}{3}u + \tfrac{1}{3}v\right)\left(\tfrac{1}{3}\right) dv\,du =$$

$$= \tfrac{1}{3}\int_0^3 \left[\tfrac{2}{3}u(3-u-0) + \tfrac{1}{6}\left((3-u)^2 - 0^2\right)\right] du = \tfrac{1}{3}\int_0^3 \left(2u - \tfrac{2}{3}u^2 + \tfrac{3}{2} - u + \tfrac{1}{6}u^2\right) du =$$

$$= \tfrac{1}{3}\int_0^3 \left(-\tfrac{1}{2}u^2 + u + \tfrac{3}{2}\right) du = \tfrac{1}{3}\left[-\tfrac{1}{6}(3^3 - 0^3) + \tfrac{1}{2}(3^2 - 0^2) + \tfrac{3}{2}(3-0)\right] =$$

$$= \tfrac{1}{3}\left[-\frac{9}{2} + \frac{9}{2} + \frac{9}{2}\right] = \boxed{3/2}$$

(2) Find the volume of the ellipsoid $x^2 + \frac{y^2}{4} + \frac{z^2}{9} \le 1$.

This problem is "almost" spherical, but we have extra constants. So first, substitute

$u = x$, $v = y/2$, $w = z/3$. Then the bound becomes

$$(u)^2 + \frac{(2v)^2}{4} + \frac{(3w)^2}{9} \le 1 \quad \longleftrightarrow \quad u^2 + \frac{4v^2}{4} + \frac{9w^2}{9} \le 1$$

$$\longleftrightarrow \quad u^2 + v^2 + w^2 \le 1.$$

So, in u, v, w space, this is a sphere (of radius 1). For the Jacobian, write

$x = u$, $y = 2v$, $z = 3w$, so $J = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6$. So:

$$\text{Volume} = \iiint_R dx\,dy\,dz = \iiint_S |J|\,du\,dv\,dw = \iiint_S 6\,du\,dv\,dw$$

$$= 6\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = 6\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\phi\,d\phi\right)\left(\int_0^1 \rho^2\,d\rho\right) =$$

$$= 6(2\pi - 0)\left(\cos(0) - \cos(\pi)\right)\left(\tfrac{1}{3}(1^3 - 0^3)\right) = 6 \cdot (2\pi)(2)\left(\tfrac{1}{3}\right) = \boxed{8\pi}$$

(Notice that since S is a sphere, we should change coordinates again to spherical coordinates in $u, v, w \longleftrightarrow \rho, \phi, \theta$.)

(3) Find the mass of a solid in the first octant bounded above by $x + y + z = 1$ with density $f(x, y, z) = x + y + z$ using

$u = x+y+z$, $v = x+y$, $w = x$.

Let's start by solving for x, y, z:

$x = w$,

$y = v - x = v - (w) = v - w$.

$z = u - x - y = u - (w) - (v-w) = u - v$

Now we'll rewrite the bounds. Since we're in the first octant, $x, y, z \geq 0$ and our other bound is $x+y+z \leq 1$. Substitution gives:

$x \geq 0 \longrightarrow (w) \geq 0 \longrightarrow \underline{w \geq 0}$

$y \geq 0 \longrightarrow (v-w) \geq 0 \longrightarrow \underline{v \geq w}$

$z \geq 0 \longrightarrow (u-v) \geq 0 \longrightarrow u \geq v$

$x+y+z \longrightarrow (w)+(v-w)+(u-v) \leq 1 \longrightarrow \underline{u \leq 1}$

Or, written compactly: $0 \leq w \leq v \leq u \leq 1$. So, the mass is:

$M = \iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_S f(u,v,w)\,|J|\,du\,dv\,dw$. To compute this, we need J:

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = 0 - 0 + 1\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = 0 - 1 = -1$$

So, $M = \iiint_R (x+y+z)\,dx\,dy\,dz = \int_0^1 \int_0^u \int_0^v u \cdot |-1| \cdot dw\,dv\,du =$

$= \int_0^1 \int_0^u u(v-0)\,dv\,du = \int_0^1 \int_0^u uv\,dv\,du = \int_0^1 \frac{1}{2}u(u^2-0^2)\,du =$

$= \frac{1}{2}\int_0^1 u^3\,du = \frac{1}{2}\cdot\frac{1}{4}\left(u^4\Big|_0^1\right) = \frac{1}{8}(1^4 - 0^4) = \boxed{1/8}$