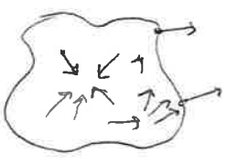
§14.1

Main Concepts

1. Vector fields
2. Divergence and curl

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1. In §12, we worked with $\vec{\nabla} f$. This is an example of a vector field, since it assigned a vector to each point (x,y) in space. E.g. if $f(x,y) = x^2 y$, then $\vec{\nabla} f(x,y) = \langle 2xy, x^2 \rangle$ tells you which direction to move at a point (x,y) to increase f the fastest. We drew these as well:

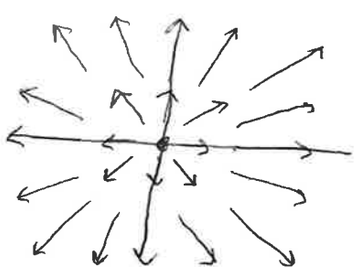


More generally, we can have vector fields with any component functions $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$, and similarly in 3D.

Examples: Sketch the following vector fields.

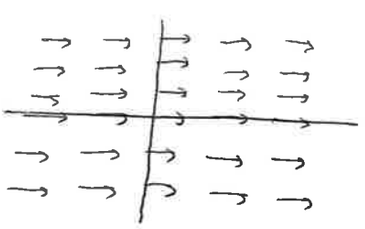
① $\vec{F}(x,y) = \langle x, y \rangle$.

This is saying that $\vec{F}$ points in the same magnitude + direction as the vector from $(0,0)$ to (x,y). So everything points "outwards" and grows:



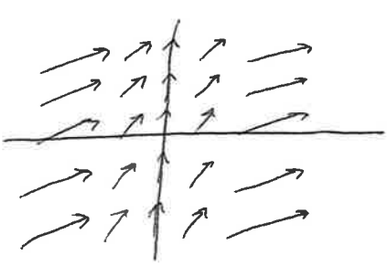
② $\vec{F}(x,y) = \langle 1, 0 \rangle$.

This is a constant vector field. No matter where you are, it points right with length 1.



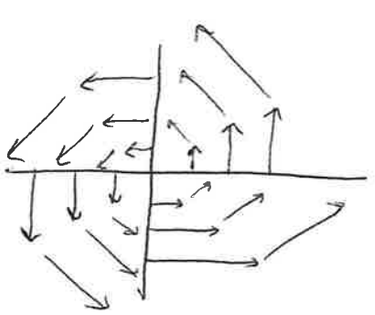
③ $\vec{F}(x,y) = \langle x^2, 1 \rangle$

Every vector has a y-component of 1. The x-component is zero when $x = 0$, but positive otherwise, and increases as $|x|$ grows.



④ $\vec{F}(x,y) = \langle -y, x \rangle$

It can be useful to draw "easy" vectors first. On the positive x-axis, $y=0$ and $x>0$ so $\vec{F}(x,y) = \langle 0, x \rangle$ points up. On the positive y-axis, $x=0$ and $y>0$ so $\vec{F}(x,y) = \langle y, 0 \rangle$ points left. Continuing gives the picture.



2 Examples ① and ④ give two interesting behaviors. ① "explodes" out of a point, and ④ "twists" around a point. We'd like to capture these facts algebraically. For both of these, it will be useful to think about $\vec{\nabla}$ again. For a scalar function f, we have $\vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle f$.

So it is useful to think of $\vec{\nabla}$ as the "vector" $\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$. Or, in 3D:

$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$. Note that this is meaningless until it's applied to a function so we actually have something to take the derivative of.

With this symbol in hand, we can define the quantities above. We have

DIVERGENCE $\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F}$ ("the divergence of $\vec{F}$"; note the dot product)

CURL $\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F}$ ("the curl of $\vec{F}$"; note the cross product).

The divergence measures how much the x-component agrees with x and similarly for y, hence measuring the "explosiveness" mentioned above. The curl instead uses the cross product, so the right-hand rule determines how much the vector field "spins" or "curls."

Examples: For each of the previous examples, compute the divergence and curl.

① $\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \cdot \langle x, y \rangle = \left(\frac{\partial}{\partial x}\right)(x) + \left(\frac{\partial}{\partial y}\right)(y) = 1 + 1 = 2$.

Notice again: we don't take any derivatives until we write $\frac{\partial}{\partial x}$(something); then we take the partial derivative of that something.

$$\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & 0 \end{vmatrix} = \left\langle \frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(y), \frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(0), \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) \right\rangle$$

$$= \langle 0-0, 0-0, 0-0 \rangle = \langle 0,0,0 \rangle = \vec{0}.$$

② $\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \cdot \langle 1, 0 \rangle = \left(\frac{\partial}{\partial x}\right)(1) + \left(\frac{\partial}{\partial y}\right)(0) = 0 + 0 = 0$

Similarly, $\operatorname{curl} \vec{F}$ will involve only derivatives of constants, so $\operatorname{curl} \vec{F} = \vec{0}$.

③ $\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = \left(\frac{\partial}{\partial x}\right)(x^2) + \left(\frac{\partial}{\partial y}\right)(\,) = 2x + 0 = 2x$.

Notice $\operatorname{div} \vec{F}$ isn't constant. I.e. near the y-axis ($x=0$), there's very little divergence, but there's more as x increases.

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & 1 & 0 \end{vmatrix} = \left\langle \frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(1),\ \frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}(0),\ \frac{\partial}{\partial x}(1) - \frac{\partial}{\partial y}(x^2) \right\rangle$$

$$= \langle 0, 0, 0 \rangle = \vec{0}$$

④ $\text{div } \vec{F} = \left(\frac{\partial}{\partial x}\right)(-y) + \left(\frac{\partial}{\partial y}\right)(x) = 0 + 0 = 0$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = \left\langle \frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x),\ \frac{\partial}{\partial z}(-y) - \frac{\partial}{\partial x}(0),\ \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) \right\rangle$$

$$= \langle 0 - 0,\ 0 - 0,\ 1 - (-1) \rangle = \langle 0, 0, 2 \rangle.$$

Note that only example ④ had curl $\vec{F} \neq \vec{0}$. This is consistent with our drawing; it's the only one twisting.

Finally, note that everything extends to 3D:

Example: For $\vec{F}(x,y,z) = \langle e^x \cos(y), -e^x \sin(y), z \rangle$, compute $\text{div } \vec{F}$ and $\text{curl } \vec{F}$.

First,

$$\text{div } \vec{F} = \vec{\nabla} \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle e^x \cos(y), -e^x \sin(y), z \rangle =$$

$$= \frac{\partial}{\partial x}\left(e^x \cos(y)\right) + \frac{\partial}{\partial y}\left(-e^x \sin(y)\right) + \frac{\partial}{\partial z}(z) =$$

$$= e^x \cos(y) - e^x \cos(y) + 1 = \boxed{1}$$

and second,

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x \cos(y) & -e^x \sin(y) & z \end{vmatrix} =$$

$$= \left\langle \frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}\left(-e^x \sin(y)\right),\ \frac{\partial}{\partial z}\left(e^x \cos(y)\right) - \frac{\partial}{\partial x}(z),\ \frac{\partial}{\partial x}\left(-e^x \sin(y)\right) - \frac{\partial}{\partial y}\left(e^x \cos(y)\right) \right\rangle$$

$$= \langle 0 - 0,\ 0 - 0,\ -e^x \sin(y) - \left(-e^x \sin(y)\right) \rangle = \langle 0, 0, 0 \rangle = \vec{0}.$$