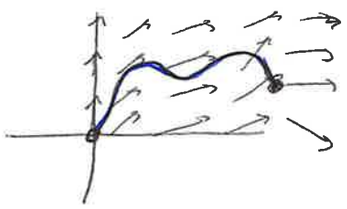
§14.2

Main Concepts

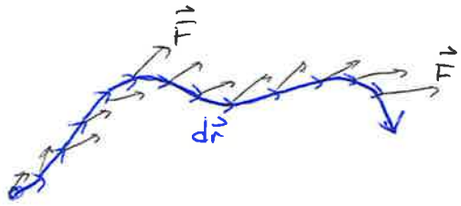
1. Line Integrals
2. Pathing

---

1. Divergence and curl gives us (two) ways differentiate a vector field. Next, we discuss one way to integrate a vector field. Inspired by physics,



we consider the case that $\vec{F}(x,y)$ represents the force exerted on a particle at a location (x,y), and let C be a path that this particle takes through space. When $\vec{F}$ and C are constants, the work done is the product (force • distance). To deal with them not being constants, we cut up C into pieces: Then we can find the work by adding up



all the pieces: $W = \int_C \vec{F} \cdot d\vec{r}$ (note the dot product).

This sounds good in principle, but how do we compute this in practice? We have a curve C, so we should parametrize it with a position function $\vec{r}(t)$, with $t \in [a,b]$. Then $d\vec{r} = \vec{r}\,'(t)\,dt$, so

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt.$$

A note on notation: Sometimes we'll write $\vec{F} = \langle P, Q \rangle$ and $\vec{r} = \langle x, y \rangle$

So $d\vec{r} = \langle dx, dy \rangle$. Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (P\,dx + Q\,dy)$$

but we should understand it via parameterization.

Examples: Compute $\int_C \vec{F} \cdot d\vec{r}$ in each of the following.

① $\vec{F}(x,y) = \langle x+y, 1 \rangle$, C is the straight line from $(0,0)$ to $(3,2)$.

First, we write a parametrization of C. We know the parametrization of a line and get:

$\vec{r}(t) = \langle 0,0 \rangle + t(\langle 3,2 \rangle - \langle 0,0 \rangle) = \langle 3t, 2t \rangle$ for $t \in [0,1]$. So

$\vec{r}\,'(t) = \langle 3, 2 \rangle$ and

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_0^1 \vec{F}(3t, 2t) \cdot \langle 3,2 \rangle\,dt = \int_0^1 \langle (3t)+(2t), 1 \rangle \cdot \langle 3,2 \rangle\,dt =$$

$$= \int_0^1 \langle 5t, 1 \rangle \cdot \langle 3,2 \rangle\,dt = \int_0^1 (15t+2)\,dt = \frac{15}{2}t^2 + 2t \Big|_0^1 = \frac{15}{2} + 2 = \boxed{\frac{19}{2}}$$

② $\vec{F}(x,y) = \langle x^2 - y^2, 2xy \rangle$, C is parametrized by $x = t^2$, $y = t^3$, $0 \le t \le 3/2$.

Here, the parametrization $\vec{r}(t) = \langle t^2, t^3 \rangle$ was given and $\vec{r}\,'(t) = \langle 2t, 3t^2 \rangle$. So,

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{3/2} \vec{F}(t^2, t^3) \cdot \langle 2t, 3t^2 \rangle\,dt = \int_0^{3/2} \langle t^4 - t^6, 2t^5 \rangle \cdot \langle 2t, 3t^2 \rangle\,dt =$$

$$= \int_0^{3/2} (2t^5 - 2t^7 + 6t^7)\,dt = \int_0^{3/2} (2t^5 + 4t^7)\,dt = \frac{1}{3}t^6 + \frac{1}{2}t^8 \Big|_0^{3/2}$$

$$= \frac{1}{3}\left(\frac{3}{2}\right)^6 + \frac{1}{2}\left(\frac{3}{2}\right)^8 = \frac{8505}{512}.$$

Or, in the other notation:

Example: Compute $\int_C (x^2\,dx + x\,dy)$ for the curve C parametrized by $x = t+1$, $y = t^3$ $0 \le t \le 1$.

Here we can either convert back to the previous notation and write $\vec{F}(x,y) = \langle x^2, x \rangle$ and $\vec{r}(t) = \langle t+1, t^3 \rangle$ $0 \le t \le 1$ or instead:

Since $x = t+1$, $dx = dt$ and $y = t^3 \rightarrow dy = 3t^2\,dt$. Then:

$$\int_C (x^2\,dx + x\,dy) = \int_0^1 \left[(t+1)^2(dt) + (t+1)(3t^2\,dt)\right] = \int_0^1 (t^2 + 2t + 1 + 3t^3 + 3t^2)\,dt =$$

$$= \int_0^1 (3t^3 + 4t^2 + 2t + 1)\,dt = \frac{3}{4}t^4 + \frac{4}{3}t^3 + t^2 + t \Big|_0^1 = \frac{3}{4} + \frac{4}{3} + 1 + 1 = \boxed{\frac{49}{12}}.$$

[2] Note that when we write $\int_C \vec{F} \cdot d\vec{r}$, we're suggesting that there's a dependence on the path C, but <u>not</u> the specific parametrization. This is <u>true</u>, but not obvious from the definition. Let's highlight this with some examples.

Examples: For $\vec{F}(x,y) = \langle y, x \rangle$ and C the upper half of the unit circle from $(1,0)$ to $(-1,0)$, compute $\int_C \vec{F} \cdot d\vec{r}$ using the given parametrizations

① $\vec{r}(t) = \langle \cos t, \sin t \rangle \quad t \in [0, \pi]$.

Same as before: $\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$ so

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi} \vec{F}(\cos t, \sin t) \cdot \langle -\sin t, \cos t \rangle \, dt = \int_0^{\pi} \langle \sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt =$$

$$= \int_0^{\pi} (\sin^2 t + \cos^2 t) \, dt = \int_0^{\pi} dt = \boxed{\pi}$$

② $\vec{r}(t) = \langle \cos(3t), \sin(3t) \rangle \quad t \in [0, \pi/3]$.

(Notice this is really the same path, just moving 3 times faster.) Here,

$\vec{r}\,'(t) = \langle -3\sin(3t), 3\cos(3t) \rangle$, so:

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/3} \vec{F}(\cos(3t), \sin(3t)) \cdot \langle -3\sin(3t), 3\cos(3t) \rangle \, dt = \int_0^{\pi/3} \langle -\sin(3t), \cos(3t) \rangle \cdot \langle -3\sin(3t), 3\cos(3t) \rangle \, dt$$

$$= \int_0^{\pi/3} \left(3\sin^2(3t) + 3\cos^2(3t)\right) dt = \int_0^{\pi/3} 3 \, dt = 3(\pi/3) = \boxed{\pi}.$$

③ $\vec{r}(t) = \langle -t, \sqrt{1-t^2} \rangle \quad t \in [-1, 1]$.

[Again, note this is still the same path. Indeed $x^2+y^2 = (-t)^2 + \left(\sqrt{1-t^2}\right)^2 = t^2 + 1 - t^2 = 1$, so this curve lies on the circle $x^2+y^2=1$; ~~and~~ $y \geq 0$ so it's the upper-half; ~~and~~ when $t=-1$, $(x,y) = (-(-1), \sqrt{1-(-1)^2}) = (1,0)$; and when $t=1$, $(x,y) = (-(1), \sqrt{1-(1)^2}) = (-1,0)$].

$\vec{r}\,'(t) = \langle -1, \frac{1}{2}(1-t^2)^{-1/2}(-2t) \rangle = \langle -1, -t(1-t^2)^{-1/2} \rangle$, so:

$$\int_C \vec{F} \cdot d\vec{r} = \int_{-1}^{1} \langle -\sqrt{1-t^2}, -t \rangle \cdot \langle -1, -t(1-t^2)^{-1/2} \rangle \, dt =$$

$$= \int_{-1}^{1} \left[(1-t^2)^{1/2} + t^2(1-t^2)^{-1/2} \right] dt = \int_{-1}^{1} (1-t^2)^{-1/2} \left((1-t^2) + t^2 \right) dt =$$

$$= \int_{-1}^{1} (1-t^2)^{-1/2} \, dt$$

$$= \arcsin(t) \Big|_{-1}^{1} = \arcsin(1) - \arcsin(-1) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \boxed{\pi}$$

As suggested, we can pick any parametrization of the curve and should get the same result.