§14.3

Main Concepts

1. Path Independence
2. Fundamental Theorem of Line Integrals
3. Finding the Potential Function

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1. Let's start with an example:

Examples: For $\vec{F} = \langle x^2, 1 \rangle$, find $\int_C \vec{F} \cdot d\vec{r}$ for the following paths.

① C is the straight line from $(-1,0)$ to $(1,0)$.

We can parametrize this by $\vec{r}(t) = \langle t, 0 \rangle$ with $t \in [-1,1]$. Then $\vec{r}\,'(t) = \langle 1, 0 \rangle$ and

$$\int_C \vec{F} \cdot d\vec{r} = \int_{-1}^{1} \langle t^2, 1 \rangle \cdot \langle 1, 0 \rangle \, dt = \int_{-1}^{1} t^2 \, dt = \frac{1}{3} t^3 \Big|_{-1}^{1} = \boxed{\frac{2}{3}}$$

② C is along the upper half of the unit circle from $(-1,0)$ to $(1,0)$.

We can use $\vec{r}(t) = \langle \cos t, \sin t \rangle$ for the unit circle again, but to ensure we go along the upper half of the circle, t has to go backwards so $t : \pi \longrightarrow 0$. Then,

$$\int_C \vec{F} \cdot d\vec{r} = \int_{\pi}^{0} \langle \cos^2 t, 1 \rangle \cdot \langle -\sin t, \cos t \rangle \, dt = \int_{\pi}^{0} \left(-\cos^2 t \sin t + \cos(t) \right) dt =$$

$$= \int_{\pi}^{0} -\cos^2(t)\sin(t)\,dt + \int_{\pi}^{0} \cos(t)\,dt$$

$$\begin{bmatrix} u = \cos(t) & du = -\sin(t)\,dt \\ t = \pi \longrightarrow & u = \cos(\pi) = -1 \\ t = 0 \longrightarrow & u = \cos(0) = 1 \end{bmatrix}$$

$$= \int_{-1}^{1} u^2 \, du + \sin(t) \Big|_{\pi}^{0} = \frac{1}{3} u^3 \Big|_{-1}^{1} + \sin(0) - \sin(\pi) =$$

$$= \frac{2}{3} + 0 + 0 = \boxed{\frac{2}{3}}$$

③ C is along the parabola $y = 1 - x^2$ between $(-1,0)$ and $(1,0)$.

We can use x itself as a parameter here. $dy = -2x\,dx$, so

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \left(x^2 dx + 1\, dy \right) = \int_{-1}^{1} \left(x^2 dx + (-2x\, dx) \right) = \int_{-1}^{1} \left(x^2 - 2x \right) dx =$$

$$= \frac{1}{3} x^3 - x^2 \Big|_{-1}^{1} = \frac{1}{3}\left((1)^3 - (-1)^3 \right) - \left((1)^2 - (-1)^2 \right) = \frac{1}{3}(2) - (0) = \boxed{\frac{2}{3}}$$

To really hammer the point home, also do:

Practice Problems: Do the same as above for C given by:

① $\vec{r}(t) = \langle 2t^3 - 1, 0 \rangle \qquad 0 \le t \le 1$

② $\vec{r}(t) = \langle t^5, t^4 - 1 \rangle \qquad -1 \le t \le 1$

③ The path along straight lines from $(-1,0)$ to $(1,2)$ to $(1,0)$. [Hint: this is the sum of two line integrals, the first with $y = x+1$ and the second with $x = 1$.]

The point is, all of these are paths from $(-1,0)$ to $(1,0)$, and all of them give the value $\int_C \vec{F}\cdot d\vec{r} = \frac{2}{3}$. Why is this happening?

First, some language. Inspired by physics, recall that if $\vec{F}$ is a force field, then $\int_C \vec{F}\cdot d\vec{r}$ is the work done, i.e. the change in energy. If $\vec{F}$ is like the example above, where $\int_C \vec{F}\cdot d\vec{r}$ depends on the endpoints, but not the path, then we're saying that the energy only depends on where you are: in other words $\vec{F}$ represents the force due to some potential energy that's conserved. Hence, we call such an $\vec{F}$ <u>conservative</u>.

Back to math: a vector field $\vec{F}$ is conservative if $\int_C \vec{F}\cdot d\vec{r}$ depends on the endpoints of C but not C itself. Can we prove any vector fields are conservative?

2 It turns out, yes! Let $f(x,y)$ be a scalar field. Then we've seen that $\vec{\nabla} f$ is a vector field. Let C be a curve parametrized by $\vec{r}(t) = \langle x(t), y(t)\rangle$ for $t\in[a,b]$. Then

$$\int_C \vec{\nabla} f \cdot d\vec{r} = \int_a^b \langle f_x, f_y\rangle \cdot \langle x', y'\rangle \, dt =$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \right) dt = \quad \text{[Multivariable Chain rule]}$$

$$= \int_a^b \left(\frac{df}{dt} \right) dt \quad \text{[Fundamental Theorem of Calculus]}$$

$$= f(\vec{r}(t))\Big|_a^b = f(\vec{r}(b)) - f(\vec{r}(a)).$$

So $\int_C \vec{\nabla} f \cdot dr$ only depends on the endpoints! I.e. $\vec{\nabla} f$ is always conservative. This explains the previous example since for $f(x,y) = \frac{1}{3}x^3 + y$,

$$\vec{\nabla} f = \langle x^2, 1\rangle = \vec{F}(x,y).$$

So $\vec{F}$ ~~from~~ above was conservative.

We've also established an important result for computation:

FUNDAMENTAL THEOREM OF LINE INTEGRALS If C is a curve from $\vec{a}$ to $\vec{b}$, then

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{b}) - f(\vec{a}).$$

[3] Finally, given $\vec{F}$ a vector field, it would be useful to know if $\vec{F} = \vec{\nabla} f$ for some f so that we can apply the previous theorem. We have the following:

THEOREM If $\vec{F}$ is defined on a simply connected region R and curl $\vec{F} = \vec{0}$, then $\vec{F} = \vec{\nabla} f$ for some f. Otherwise $\vec{F}$ is not conservative.

The condition of R being simply connected is necessary but technical; we will often ignore it [for those interested, this roughly says that R has no "tunnels" running all the way through it].

Examples: Check if the following vector fields are conservative.

① $\vec{F}(x,y) = \langle x, y \rangle$

We computed curl $\vec{F} = \vec{0}$ in §14.1. So $\vec{F}$ is conservative

② $\vec{F}(x,y) = \langle 1, 0 \rangle$.

Again, we did this previously and curl $\vec{F} = 0$. So $\vec{F}$ is conservative

③ $\vec{F}(x,y) = \langle -y, x \rangle$.

We did this previously and found curl $\vec{F} = \langle 0, 0, 2 \rangle$, so this is <u>not</u> conservative.

④ $\vec{F}(x,y,z) = \langle e^x \cos(y), -e^x \sin(y), z \rangle$.

This is in 3D, but we found curl $\vec{F} = \vec{0}$, so it is also conservative.

Okay, so for ①, ②, and ④, $\vec{F} = \vec{\nabla} f$ for some f. But what's f? The idea is as follows:

① Write $\vec{F} = \langle P, Q \rangle$ (or $\langle P, Q, R \rangle$ for 3D).

② Take the x-partial antiderivative of P.

- Don't forget $+C$, but since y is a constant, C can depend on y.

③ Differentiate your result with respect to y.

④ This should equal Q. Set it equal to Q to solve for C.

In other words, we have $\langle P, Q \rangle = \vec{F} = \vec{\nabla} f = \langle f_x, f_y \rangle$. So

$f = "\int P \, dx"$ but we need to solve for C, so differentiate to get

$Q = f_y = \frac{\partial}{\partial y} \int P \, dx$ and solve.

Examples: Find f such that $\vec{\nabla} f = \vec{F}$ for the given $\vec{F}$.

① $\vec{F}(x,y) = \langle x, y \rangle$.

We want f so that $f_x = x$ and $f_y = y$. From the first equation, we get

$f(x,y) = \int x\,dx = \frac{1}{2}x^2 + C(y)$. Differentiating with respect to y gives:

$y = f_y = 0 + C'(y)$. So $C'(y) = y$. Integrating again gives

$C(y) = \int C'(y)\,dy = \int y\,dy = \frac{1}{2}y^2 + D$. So, overall,

$f(x,y) = \frac{1}{2}x^2 + C(y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + D$. We're just looking for any f, so we might as well choose $D = 0$, giving $\boxed{f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2}$

(2) $\vec{F}(x,y) = \langle 1, 0 \rangle$.

Again, we want $f_x = 1$ and $f_y = 0$. The first gives $f(x,y) = \int f_x\,dx = \int 1\,dx = x + C(y)$.

Then differentiating: $0 = f_y = C'(y)$ so $C(y)$ is a constant: we can again choose 0 for ease, giving $\boxed{f(x,y) = x}$

(3) $\vec{F}(x,y) = \langle 2xy, x^2 + 3y^2 \rangle$

[First, check curl $\vec{F} = \vec{0}$ so that $\vec{F} = \vec{\nabla} f$ is even possible].

We want $f_x = 2xy$ and $f_y = x^2 + 3y^2$. From the first,

$f(x,y) = \int f_x\,dx = \int 2xy\,dx = x^2y + C(y)$. Differentiating in y:

$x^2 + 3y^2 = f_y = x^2 + C'(y) \longrightarrow C'(y) = 3y^2$. Integrating gives

$C(y) = \int C'(y)\,dy = \int 3y^2\,dy = y^3$. So $\boxed{f(x,y) = x^2y + y^3}$

(4) $\vec{F}(x,y,z) = \langle e^x\cos(y), -e^x\sin(y), z \rangle$.

There are 3 components, but the idea is the same. From $f_x = e^x\cos(y)$ we get:

$f(x,y,z) = \int e^x\cos(y)\,dx = e^x\cos(y) + C(y,z)$. Note that the constant can depend on both y and z. Taking $\frac{\partial}{\partial y}$ gives

$-e^x\sin(y) = f_y = -e^x\sin(y) + C_y \longrightarrow C_y = 0$. So, integrating gives

$C(y,z) = \int C_y\,dy = \int 0\,dy = D(z)$. I.e. $f(x,y,z) = e^x\cos(y) + D(z)$.

Taking $\frac{\partial}{\partial z}$ gives

$z = f_z = D'(z)$ and integrating gives $D(z) = \frac{1}{2}z^2$.

So $\boxed{f(x,y,z) = e^x\cos(y) + \frac{1}{2}z^2}$.