§14.4

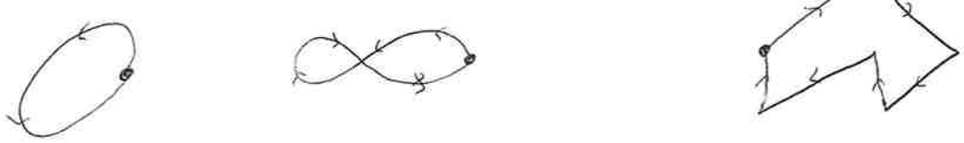
Main Concepts

1. Green's Theorem
2. Vector form

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1. Green's theorem is a fundamental result relating line integrals and double integrals.

First, some language: we'll say a curve C is a <u>closed curve</u> if the two endpoints are the same. Some examples:



A curve C is <u>simple</u> if it doesn't intersect itself (so not the figure eight, but the other two are okay). If C is a simple closed curve, then we can see that it's also the boundary of a region R.



Finally, since C is a curve, if $\vec{F}$ is a vector field, we can compute $\int_C \vec{F}\cdot d\vec{r}$. If C is a <u>closed and oriented counterclockwise</u>, we will write

$$\oint_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r}.$$

Now we can state Green's theorem:

GREEN'S THEOREM: Suppose C is a simple closed curve that is the boundary of a region R in the plane. Let $\vec{F} = \langle P, Q \rangle$ be a vector field. Then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

or, equivalently:

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

This lets us convert back-and-forth between double integrals and line integrals, evaluating whichever is easier.

— Examples: Evaluate $\oint_C \vec{F}\cdot d\vec{r}$ both directly and using Green's theorem.

① $\vec{F}(x,y) = \langle -y, x\rangle$, C is the full unit circle.

Directly: we need to parametrize C (counterclockwise). One way is to use $\vec{r}(t) = \langle \cos t, \sin t\rangle$ for $t\in[0,2\pi]$. Then

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\cos t, \sin t)\cdot\langle -\sin t, \cos t\rangle\, dt = \int_0^{2\pi}\langle -\sin t, \cos t\rangle\cdot\langle -\sin t, \cos t\rangle\, dt =$$

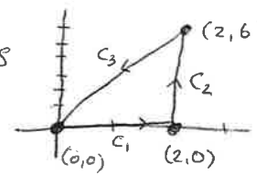
$$= \int_0^{2\pi}(\sin^2 t + \cos^2 t)\, dt = \int_0^{2\pi} dt = \boxed{2\pi}.$$

Green: We get an integral over the unit disk, which we do in polar:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right) dA = \int_0^{2\pi}\int_0^1 (1-(-1))\, r\, dr\, d\theta = 2(2\pi)\left[\tfrac{1}{2}(1)^2 - \tfrac{1}{2}(0)^2\right]$$

$$= 4\pi\left(\tfrac{1}{2}\right) = \boxed{2\pi}.$$

② $\vec{F}(x,y) = \langle xy, 1\rangle$, C is the triangle with vertices $(0,0)$, $(2,0)$, and $(2,6)$.



Directly: We need to parametrize C (counterclockwise), which splits into 3 separate integrals. On the first part, $y=0$ and $x: 0\to 2$. So $dy = 0\,dx$ and

$$\int_{C_1}\vec{F}\cdot d\vec{r} = \int_{C_1}(xy\,dx + 1\,dy) = \int_0^2 (x(0)\,dx + 1(0)) = \int_0^2 0\,dx = 0.$$

On the second part, $x=2$, $y: 0\to 6$, $dx = 0\,dy$, and:

$$\int_{C_2}\vec{F}\cdot d\vec{r} = \int_{C_2}(xy\,dx + 1\,dy) = \int_0^6 \left[(2)y(0\,dy) + 1\,dy\right] = \int_0^6 1\,dy = 6-0 = 6$$

For the third part, we need to parametrize the line. One way to do so is with $\vec{r}(t) = \langle 2,6\rangle + t(\langle 0,0\rangle - \langle 2,6\rangle) = \langle 2-2t, 6-6t\rangle$ with $t\in[0,1]$. Then

$$\int_{C_3}\vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(2-2t, 6-6t)\cdot\langle -2,-6\rangle\, dt = \int_0^1 \langle (2-2t)(6-6t), 1\rangle\cdot\langle -2,-6\rangle\, dt =$$

$$= \int_0^1 \left[2\cdot 6(1-t)^2\cdot(-2) + (1)(-6)\right] dt = \int_0^1 \left[-24(1-2t+t^2) - 6\right] dt =$$

$$= \int_0^1 (-24t^2 + 48t - 30)\, dt = -8t^3 + 24t^2 - 30t\Big|_0^1 = -8 + 24 - 30 = -14$$

Putting it all together: $\oint_C \vec{F}\cdot d\vec{r} = \int_{C_1}\vec{F}\cdot d\vec{r} + \int_{C_2}\vec{F}\cdot d\vec{r} + \int_{C_3}\vec{F}\cdot d\vec{r} = (0)+(6)+(-14) = \boxed{-8}$

Green: We now want to integrate on the interior of the triangle. The top will be one of the bounds, so we should find an equation for that line. It has slope $\frac{6}{2} = 3$ and passes through $(0,0)$, so it's $y = 3x$. Then, we get:

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^2 \int_0^{3x} \left(\frac{\partial}{\partial x}(1) - \frac{\partial}{\partial y}(xy)\right) dy\,dx = \int_0^2 \int_0^{3x} (0 - x)\,dy\,dx = \int_0^2 -xy\Big|_{y=0}^{3x} dx =$$

$$= \int_0^2 -x(3x-0)\,dx = \int_0^2 -3x^2\,dx = -x^3\Big|_0^2 = -(2)^3 + (0)^3 = \boxed{-8}$$

Note especially in the 2nd example how much easier the double integral was, and thus one advantage of Green's theorem.

Green's theorem also works in reverse, of course. One particularly useful case is to compute the area of a region R. If C is the boundary of this region and $\vec{F}(x,y) = \langle -y, x\rangle$, then:

$$\text{Area}(R) = \iint_R dA = \iint_R \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right)\cdot\frac{1}{2}\,dA = \frac{1}{2}\iint_R \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right) dA =$$

$$= \frac{1}{2}\oint_C \langle -y, x\rangle \cdot d\vec{r} = \frac{1}{2}\oint_C (-y\,dx + x\,dy).$$

Similarly, we can derive: $\text{Area}(R) = \oint_C x\,dy = \oint_C (-y\,dx)$ as well.

Example: Use a line integral to find the area of the ellipse $x^2 + \frac{y^2}{4} \le 1$.

To use a line integral, we need to parametrize the boundary C: $x^2 + y^2/4 = 1$. This can be done by $\vec{r}(t) = \langle \cos(t), 2\sin(t)\rangle$ for $t \in [0, 2\pi]$. Then,

$$\text{Area} = \frac{1}{2}\oint_C (-y\,dx + x\,dy) = \frac{1}{2}\int_0^{2\pi} \left[-(2\sin(t))(-\sin(t)\,dt) + (\cos(t))(2\cos(t)\,dt)\right] =$$

$$= \frac{1}{2}\int_0^{2\pi} (2\sin^2(t) + 2\cos^2(t))\,dt = \int_0^{2\pi} dt = \boxed{2\pi}$$

Note that before, to find the area, we would have used the coordinate transformation $u = x$ and $v = y/2$, but this approach avoids the Jacobian & polar coordinates.

2 Finally, we remark on another way to see Green's theorem without using components (P, Q, etc.). Let $\vec{F} = \langle P, Q\rangle$ as before. Then

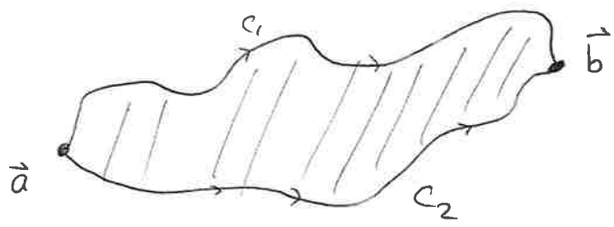
$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix} = \left\langle \frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(Q), \frac{\partial}{\partial z}(P) - \frac{\partial}{\partial x}(0), \frac{\partial}{\partial x}(Q) - \frac{\partial}{\partial y}(P)\right\rangle$$

$$= \left\langle 0-0, 0-0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle \quad \text{since } P, Q \text{ only depend on } x, y,$$

and not z. So $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is the z-component of $\text{curl}\,\vec{F}$, i.e. it's $(\text{curl}\,\vec{F})\cdot\hat{k}$. We can rewrite Green's theorem as

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (\text{curl}\,\vec{F})\cdot\hat{k}\,dA$$

This is good, because it justifies something we said earlier. Namely, suppose $\operatorname{curl}\vec{F} = 0$ and C_1, C_2 are different paths with the same endpoints $\vec{a}$, $\vec{b}$.



Then we can take the overall path $C = C_2 + \overline{C_1}$, where $\overline{C_1}$ is the same as C_1, but backwards. This gives a counterclockwise closed path bounding a region R, so:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (\operatorname{curl}\vec{F})\cdot\hat{k}\, dA = \iint_R 0\, dA = 0$$

since $\operatorname{curl}\vec{F} = 0$. But $\oint_C \vec{F}\cdot dr = \int_{C_2}\vec{F}\cdot d\vec{r} + \int_{\overline{C_1}}\vec{F}\cdot d\vec{r} = \int_{C_2}\vec{F}\cdot d\vec{r} - \int_{C_1}\vec{F}\cdot d\vec{r}$.

Plugging this in gives $\int_{C_2}\vec{F}\cdot dr - \int_{C_1}\vec{F}\cdot d\vec{r} = 0$ or

$$\int_{C_2}\vec{F}\cdot d\vec{r} = \int_{C_1}\vec{F}\cdot d\vec{r}.$$

In other words, $\int \vec{F}\cdot dr$ is <u>path-independent</u>, so $\vec{F}$ is conservative as claimed.