Section 1.1

Main Concepts

1. Limits
2. One-sided limits
3. Plugging in

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1. The most fundamental idea in the development of calculus is that of a limit. Specifically, we ask the question: for a function $f(x)$, what happens as x approaches (but doesn't reach) a number "a"?

- Examples

① Let $f(x) = x^2$ and $a = 2$. We're wondering what happens if x is "near" 2.

$f(1) = 1^2 = 1$

$f(1.5) = 1.5^2 = 2.25$

$f(1.9) = 1.9^2 = 3.61$

$f(1.99) = 1.99^2 = 3.9601$

$f(1.99999) = 1.99999^2 = 3.9999600001$

We can see, as $x \longrightarrow 2$, $f(x) \longrightarrow 4$. Of course, this isn't so surprising since $f(2) = 2^2 = 4$.

② Let $f(x) = \frac{x^2-1}{x-1}$, $a = 1$. This time, we can't skip to the end and plug in 1, since $f(1) = \frac{1^2-1}{1-1} = \frac{0}{0}$ is indeterminate. But we can use the other approach:

$f(1.1) = 2.1$

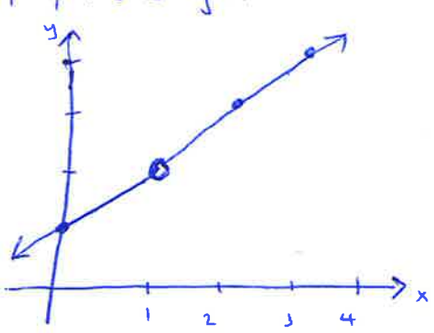
$f(1.01) = 2.01$

$f(1.00001) = 2.00001$

It looks like, as $x \to 1$, $f(x) \longrightarrow 2$. Can we explain this? Yes, by factoring we see that $f(x) = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1$ (for $x \neq 1$).

Then if we plug in $x=1$, we'd get $f(x) \longrightarrow (1)+1 = 2$.

Graphically, we have:



(Notice the gap at $x=1$).

We see, in the examples, some strategies: Plug in values of x near a, and keep getting closer until you can tell what's happening. NOTE: Patterns can be misleading, so one goal of this class is to develop more robust strategies.

1 Once we figure out the value that $f(x)$ approaches, we write it like so:

$$\lim_{x \to a} f(x).$$

In other words, in example ① we found $\lim_{x \to 2} x^2 = 4$ and in example ② we found $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$.

2 If we look back at our examples, in example ① we only plugged in values of $x < a$, and in example ② we plugged in $x > a$. This can be misleading.

— EXAMPLE

Let $f(x) = \begin{cases} x^2 + 1 & \text{if } x \geq 1 \\ 2 - x & \text{if } x < 1 \end{cases}$. What is $\lim_{x \to 1} f(x)$?

If we try, as before, plugging in values near 1 (on both sides), we get:

$f(0) = 2 - 0 = 2$

$f(0.5) = 2 - 0.5 = 1.5$

$f(0.9) = 2 - 0.9 = 1.1$

$f(0.99) = 2 - 0.99 = 1.01$

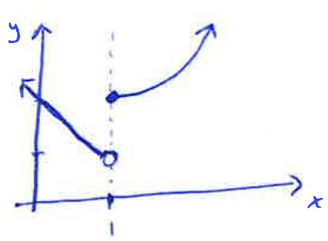
$f(2) = 2^2 + 1 = 5$

$f(1.5) = 1.5^2 + 1 = 3.25$

$f(1.1) = 1.1^2 + 1 = 2.21$

$f(1.01) = 1.01^2 + 1 = 2.0201$

The "lower" numbers approach 1, while the "upper" numbers approach 2. In other words, $f(x)$ doesn't approach any single number. Again, we can understand this graphically:



(Notice the jump at $x = 1$)

In cases like this, we can still say something. The limit as $x \to 1$ does not exist, but the "left-hand" limit and the "right-hand" limit do exist. In symbols, we'll write:

$\lim_{x \to a^-} f(x) =$ "Left-hand" limit or Limit from below

$\lim_{x \to a^+} f(x) =$ "Right-hand" limit or Limit from above.

So, in the example above, we showed

$$\lim_{x \to 1^-} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to 1^+} f(x) = 2.$$

We can summarize our observations into a theorem:

THEOREM If $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist and $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} f(x) = \lim_{x \to a^+} f(x)$.

3 Finally, a note on plugging in: this won't always work, but can give you some idea of what to expect. Specifically, if plugging in gives a concrete number, we're more likely to believe the limit exists and equals that number. If plugging in gives $\frac{0}{0}$, we have some hope (maybe the "0"s cancel like in $\frac{x^2-1}{x-1}$). If it gives $\frac{\text{constant}}{0}$, we have far less hope.

EXAMPLE

Does $\lim_{x \to 0} \frac{1}{x}$ exist? What is it?

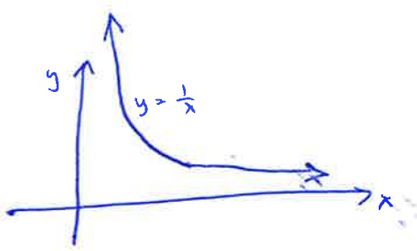
As before, we plug in to get a guess: $\frac{1}{(0)}$ is undefined so probably no luck.

Let's confirm: for $f(x) = \frac{1}{x}$,

$f(1) = \frac{1}{1} = 1$

$f(0.1) = \frac{1}{0.1} = 10$

$f(0.01) = \frac{1}{0.01} = 100$



It seems to just be getting bigger, as expected from the graph.