Section 1.2

Main Concepts

1. Formality (and why?)
2. Definition of the limit

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1. Consider the following example (from the textbook):

Example: Guess the value of $\lim_{x \to 0} \left(x^2 - \frac{\cos(x)}{10000} \right)$.

As we've been doing, let's approach $a = 0$: for $f(x) = x^2 - \frac{\cos(x)}{10000}$,

$f(1) = 0.99995\ldots$

$f(0.5) = 0.24991\ldots$

$f(0.1) = 0.00990\ldots$

$f(0.01) = 0.000000005\ldots$

So, we guess $\lim_{x \to 0} f(x) = 0$

On the other hand, we've seen plugging in works sometimes. This gives

$f(0) = (0)^2 - \frac{\cos(0)}{10000} = 0 - \frac{1}{10000} = -\frac{1}{10000}$ which isn't 0...

The problem we're running into is that we're not being careful enough. We're saying x "approaches" zero, but how close does it have to get?

Let's reconsider the example of $f(x) = x^2$ as $x \longrightarrow 2$. We said $f(x)$ approaches 4, but let's try to be more precise: suppose the distance between x and 2 is less than 0.01. In other words, $1.99 < x < 2.01$. Squaring gives $1.99^2 < x^2 < 2.01^2$, i.e. $3.9601 < x^2 < 4.0401$. So $f(x) = x^2$ is indeed "close" to 4. How close? — the distance is at most 0.0401 since $-0.0399 < x^2 - 4 < 0.0401$.

This is getting closer to a proper definition. To finish it, we notice that the question we posed is "backwards" — we said if the x-distance is small (< 0.01), then the y-distance is small (< 0.0401). But ideally, we'd want the y-distance to be as small as we want, as long as we choose the right x-distance.

For example, can we make it accurate to $\frac{1}{1000000}$? I.e. can we make $4 - \frac{1}{10^6} < x^2 < 4 + \frac{1}{10^6}$ as long as $2 - \delta < x < 2 + \delta$ for some value of δ? Sure, $\delta = \frac{1}{10^7}$ works:

* Footnote: this is the greek letter (lowercase) delta.

If $2 - \frac{1}{10^7} < x < 2 + \frac{1}{10^7}$, then $\left(2 - \frac{1}{10^7}\right)^2 < x^2 < \left(2 + \frac{1}{10^7}\right)^2$, and expanding gives

$$4 - \frac{4}{10^7} + \frac{1}{10^{14}} < x^2 < 4 + \frac{4}{10^7} + \frac{1}{10^{14}}.$$ I.e.

$3.9999996000000 1 < x^2 < 4.0000004000000 1$ and in particular,

$3.999999 \quad < x^2 < 4.000001$ as we hoped.

2 This gives us the proper definition of the limit.

* Another Greek letter: epsilon

DEFINITION We say $\lim_{x \to a} f(x) = L$ if, for every $\varepsilon > 0$, there is some $\delta > 0$ such that $a - \delta < x < a + \delta$ guarantees that

$$L - \varepsilon < f(x) < L + \varepsilon.$$

EXAMPLE: I claim $\lim_{x \to 0} x^2 = 0$.

To show this, let $\varepsilon > 0$. Then choose $\delta = \sqrt{\varepsilon}$. Pick some x such that

$0 - \delta < x < 0 + \delta$ i.e $-\sqrt{\varepsilon} < x < \sqrt{\varepsilon}$.

Squaring both sides gives $0 \leq x^2 < \varepsilon$, and $-\varepsilon < 0$, so

$-\varepsilon < x^2 < \varepsilon$, which is what we wanted to show.