Section 1.3

Main Concepts

1. Main Limit Theorem and Consequences
2. Locality
3. Squeeze Theorem

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1. The formality of section 1.2 is nice because it gives us sure footing, but it's a very hard definition to use practically. We'd like to have a more practical tool, even if it doesn't cover all cases. We have such a tool:

Main Limit Theorem: Let a and k be numbers, n be a positive integer, and f and g functions such that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist. Then:

(I) $\lim_{x\to a} k = k$

(II) $\lim_{x\to a} x = a$

(III) $\lim_{x\to a} [kf(x)] = k\left[\lim_{x\to a} f(x)\right]$

(IV) $\lim_{x\to a} [f(x)+g(x)] = \left[\lim_{x\to a} f(x)\right] + \left[\lim_{x\to a} g(x)\right]$

(V) $\lim_{x\to a} [f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]$

(VI) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ as long as $\lim_{x\to a} g(x) \neq 0$

(VII) $\lim_{x\to a} [f(x)^n] = \left[\lim_{x\to a} f(x)\right]^n$

(VIII) $\lim_{x\to a} \left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim_{x\to a} f(x)}$ as long as $\lim_{x\to a} f(x) > 0$ when n is even.

All of these conclusions also hold for one-sided limits (from above or below).

Before we start using this tool, let's try to understand what it's saying more intuitively.

Part (I) says that if $f(x) = k$ is a constant function, then as x approaches a, $f(x)$ approaches k.

Part (II) says that if x approaches a, then x approaches a. ($=f(x)$)

Part (III) says that if $f(x)$ approaches L, then if you multiply $f(x)$ by a constant k, then the new values approach $k \cdot L$.

Part (IV) involves two functions. It says that if $f(x)$ is near L and $g(x)$ is near M, then $f(x)+g(x)$ is near $L+M$. Parts (V) – (VIII) are similar. Let's see why this is so useful:

EXAMPLES: Compute the following limits.

① $\lim_{x\to 2}(4x^3)$

We'll keep applying the Main Limit Theorem (MLT) and indicate which part we use:

$$\lim_{x\to 2}(4x^3) \overset{(III)}{=} 4\lim_{x\to 2}(x^3) \overset{(VII)}{=} 4\left(\lim_{x\to 2}x\right)^3 \overset{(II)}{=} 4(2)^3 = 4\cdot 8 = 32$$

② $\lim_{x\to 1}(2x+x^7)$

Same idea:

$$\lim_{x\to 1}(2x+x^7) \overset{(IV)}{=} \left(\lim_{x\to 1}2x\right)+\left(\lim_{x\to 1}x^7\right) \overset{(III)\text{ and }(VII)}{=} 2\left(\lim_{x\to 1}x\right)+\left(\lim_{x\to 1}x\right)^7 \overset{(II)}{=} 2(1)+(1)^7 = 3$$

③ $\lim_{x\to 0}\dfrac{x^2-3x+5}{x+1}$

Same idea:

$$\lim_{x\to 0}\frac{x^2-3x+5}{x+1} \overset{(VI)}{=} \frac{\lim_{x\to 0}(x^2-3x+5)}{\lim_{x\to 0}(x+1)} \overset{(IV)\text{ repeatedly}}{=} \frac{\lim_{x\to 0}(x^2)+\lim_{x\to 0}(-3x)+\lim_{x\to 0}(5)}{\lim_{x\to 0}(x)+\lim_{x\to 0}(1)}$$

$$\overset{(III)\text{ and }(VII)}{=} \frac{\left(\lim_{x\to 0}x\right)^2+(-3)\left(\lim_{x\to 0}x\right)+\lim_{x\to 0}(5)}{\lim_{x\to 0}x+\lim_{x\to 0}1} \overset{(I)\text{ and }(II)}{=} \frac{(0)^2+(-3)(0)+(5)}{(0)+(1)} = \frac{5}{1} = 5.$$

Notice in each of these examples, we got the same answers we would get by simply plugging in $x=a$: in ① we got $4(2)^3=32$, in ② we got $2(1)+(1)^7=3$ and in ③ we got $\dfrac{(0)^2-3(0)+5}{(0)+1}$. We can also see why — the MLT allowed us to break apart sums, products, coefficients, and fractions until the only limits left were $\left(\lim_{x\to a}x\right)$, or $\left(\lim_{x\to a}\text{constant}\right)$. As long as $f(x)$ has this form, we can always do this. We're convinced of the following:

THEOREM If $f(x)$ is a polynomial, then $\lim_{x\to a}f(x)=f(a)$. If $f(x)$ is a fraction where both numerator and denominator are polynomials, and a is in the domain of f, then $\lim_{x\to a}f(x)=f(a)$.

Directly using this theorem can save time and steps:

EXAMPLE: Compute $\lim_{x\to 1} \dfrac{4x^3-7x^2+2}{x^4+x^3-10}$

Both numerator & denominator are polynomials, so

$$\lim_{x\to 1} \frac{4x^3-7x^2+2}{x^4+x^3-10} = \frac{4(1)^3-7(1)^2+2}{(1)^4+(1)^3-10} = \frac{4-7+2}{1+1-10} = \frac{-1}{-8} = 1/8$$

Finally a note: why didn't we say anything about subtraction in the MLT? Answer: we didn't need to, because we can use what we already know:

$$\lim_{x\to a}\left[f(x)-g(x)\right] = \lim_{x\to a}\left[f(x)+(-g(x))\right] \overset{\text{(IV)}}{=} \lim_{x\to a} f(x) + \lim_{x\to a}\left[(-1)g(x)\right]$$

$$\overset{\text{(III)}}{=} \lim_{x\to a} f(x) + (-1)\lim_{x\to a} g(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x).$$

2 Let's recall one of our first examples: $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$. When we solved this, we plugged in numbers close to 1 and saw the output was close to 2. We also convinced ourselves this is correct by factoring: $\dfrac{x^2-1}{x-1} = \dfrac{(x-1)(x+1)}{x-1} = x+1$, which holds for $x \neq 1$. However, neither the MLT or our second theorem justifies this: we need another tool.

SUBSTITUTION THEOREM If $f(x)=g(x)$ except possibly at $x=a$, then either $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ do not exist, or they both exist and are equal. The same is true if $f(x)=g(x)$ in an interval containing a (except possibly at $x=a$).

This is now the justification: except at $x=1$, we have $\dfrac{x^2-1}{x-1} = x+1$. So, since $\lim_{x\to 1}(x+1) = 2$ exists, so does $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$, and it also equals 2.

We can do this in other examples, too:

EXAMPLES: Compute the following limits.

① $\lim_{x\to 5} \dfrac{x^2-25}{x^2-4x-5}$.

We'd like to use the MLT, but the denominator would be $(5)^2-4(5)-5 = 25-20-5 = 0$, so it doesn't apply. But we can factor, so we get

$$\lim_{x\to 5} \frac{x^2-25}{x^2-4x-5} = \lim_{x\to 5} \frac{(x-5)(x+5)}{(x-5)(x+1)} \overset{\star}{=} \lim_{x\to 5} \frac{x+5}{x+1} = \frac{(5)+5}{(5)+1} = \frac{10}{6} = \frac{5}{3}.$$

Where the ★'d step is justified by the Substitution Theorem.

(2) $\lim_{x\to 3} \dfrac{\sqrt{x+1}-2}{x^2-9}$

Again, it would be nice to use MLT, but the denominator is $(3)^2-9=9-9=0$. We can factor the denominator: $x^2-9=(x-3)(x+3)$. However, we can't factor the numerator — to simplify it, we'd be best off multiplying by the conjugate $\sqrt{x+1}+2$. So overall,

$$\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x^2-9}\cdot\left(\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}\right) \overset{(ST)}{=} \lim_{x\to 3} \frac{(\sqrt{x+1})^2-(2)^2}{(x-3)(x+3)(\sqrt{x+1}+2)} = \lim_{x\to 3} \frac{x+1-4}{(x-3)(x+3)(\sqrt{x+1}+2)}$$

$$= \lim_{x\to 3} \frac{x-3}{(x-3)(x+3)(\sqrt{x+1}+2)} \overset{(ST)}{=} \lim_{x\to 3} \frac{1}{(x+3)(\sqrt{x+1}+2)} \overset{(IV)}{=} \frac{\lim_{x\to 3} 1}{\lim_{x\to 3}[(x+3)(\sqrt{x+1}+2)]}$$

$$\overset{(I),(V)}{=} \frac{1}{\left[\lim_{x\to 3}(x+3)\right]\left[\lim_{x\to 3}(\sqrt{x+1}+2)\right]} \overset{(IV)}{=} \frac{1}{\left[\lim_{x\to 3} x+\lim_{x\to 3} 3\right]\left[\lim_{x\to 3}\sqrt{x+1}+\lim_{x\to 3} 2\right]}$$

$$\overset{(I),(II),(VIII)}{=} \frac{1}{(3+3)\left(\sqrt{\lim_{x\to 3}(x+1)}+2\right)} \overset{(I),(II),(IV)}{=} \frac{1}{(3+3)(\sqrt{3+1}+2)} = \frac{1}{6(\sqrt{4}+2)} = \frac{1}{6(2+2)} = \frac{1}{6\cdot 4} = \frac{1}{24}$$

where each step is justified by MLT or the Substitution Theorem (ST).

3 Finally, we have one more tool worth mentioning. Notice that MLT & ST only help with "algebraic" problems. We have no way to do anything with e^x, $\log(x)$, $\sin(x)$, etc. The next theorem can sometimes help with this.

SQUEEZE THEOREM If $f(x)\le g(x)\le h(x)$ for all x in some interval containing a, except possibly at a itself, and $\lim_{x\to a} f(x)=L=\lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x)=L$.

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EXAMPLE: Later in the course, we'll show $x-\frac{x^3}{6}\le \sin(x)\le x$ for $0\le x\le 0.1$.

So, $1-\frac{x^2}{6}\le \frac{\sin(x)}{x}\le 1$ for $0<x\le 0.1$. But if $-0.1\le x<0$, then

$\frac{\sin(x)}{x}=\frac{-\sin(x)}{-x}=\frac{\sin(-x)}{(-x)}$ and $0<-x\le 0.1$ so $1-\frac{(-x)^2}{6}\le\frac{\sin(-x)}{(-x)}\le 1$, i.e.

$$1-\frac{x^2}{6}\le\frac{\sin x}{x}\le 1$$

for all $x\in[-0.1,0.1]$ except 0. By the MLT:

- $\lim_{x\to 0}\left(1-\frac{x^2}{6}\right)=1-\frac{(0)^2}{6}=1$ and $\lim_{x\to 0}(1)=1$.

These are the same, so by the Squeeze Theorem, $\lim_{x\to 0}\frac{\sin(x)}{x}=1$ as well.