§1.4

Main Concepts

[1] Continuity of Trig Functions

[2] Special limits

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[1] We've seen so far that, if $f(x)$ is a rational function and a is in the domain, then $\lim_{x\to a} f(x) = f(a)$. That is, we can compute limits just by plugging in. The same is true for trig functions:

Theorem: If $f(x)$ is a trig function ($\sin, \cos, \tan, \csc, \sec, \cot$), and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$.

EXAMPLES: Compute the following limits:

① $\lim_{x\to 0} \sin(x)$

Directly using the theorem, $\lim_{x\to 0} \sin(x) = \sin(0) = 0$.

② $\lim_{x\to \pi/4} \tan(x)$

Same thing: $\lim_{x\to \pi/4} \tan(x) = \tan(\pi/4) = 1$.

③ $\lim_{x\to \pi/2} \tan(x)$.

In this case we have no tools since $\pi/2$ is not in the domain of $\tan(x)$.

④ $\lim_{x\to 0} \frac{\cos(x)}{x^2+1}$.

By the Main Limit Theorem, this is $\frac{\lim_{x\to 0} \cos(x)}{\lim_{x\to 0} (x^2+1)}$ as long as both limits exist and the denominator is not zero. But $\lim_{x\to 0} \cos(x) = \cos(0) = 1$ and $\lim_{x\to 0} (x^2+1) = 0^2+1 = 1$, so

$$\lim_{x\to 0} \frac{\cos(x)}{x^2+1} = \frac{1}{1} = 1.$$

We see that in combination with the MLT, this theorem is a powerful tool, and lets us compute more complicated limits than before.

[2] Nonetheless, we cannot yet compute every limit involving trig functions. Here are two useful limits that we will not prove for now: they can be proven using the squeeze theorem.

FACTS

- $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$
- $\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$

Note that we cannot prove these by using the MLT since we'd get $\frac{0}{0}$ in both cases. This also lets us solve more problems:

EXAMPLES:

① $\lim_{x\to 0} \frac{\sin(x)}{2x}$

By MLT, we can pull out the $\frac{1}{2}$: $\lim_{x\to 0} \frac{\sin(x)}{2x} = \frac{1}{2}\lim_{x\to 0} \frac{\sin(x)}{x} = \frac{1}{2}(1) = \frac{1}{2}$.

② $\lim_{x\to 0} \frac{\sin^2(x)}{4x}$.

Same idea; write $\sin^2(x) = \sin(x)\cdot\sin(x)$ and split up the limit:

$$\lim_{x\to 0} \frac{\sin^2(x)}{4x} = \frac{1}{4}\lim_{x\to 0} \frac{\sin(x)\sin(x)}{x} = \frac{1}{4}\left(\lim_{x\to 0}\sin(x)\right)\left(\lim_{x\to 0}\frac{\sin(x)}{x}\right) = \frac{1}{4}(\sin(0))\cdot(1)$$

$$= \frac{1}{4}(0)(1) = 0.$$