§1.5

Main Concepts

1. Limit at $\pm\infty$
2. ~~Vertical~~ Horizontal asymptotes
3. ~~Horizontal~~ Vertical asymptotes

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1. When we compute $\lim_{x\to a} f(x)$, we ask, "what happens to $f(x)$ as x gets closer and closer to a?" We can do the same thing when $a=\infty$.

EXAMPLE: Compute $\lim_{x\to\infty} \frac{1}{x}$.

We make x "closer and closer to infinity" by choosing increasing values for x:

$f(1) = \frac{1}{1} = 1$

$f(2) = \frac{1}{2} = 0.5$

$f(10) = \frac{1}{10} = 0.1$

$f(100) = \frac{1}{100} = 0.01$

$f(1000000) = \frac{1}{1000000} = 0.000001$

It seems clear that $\lim_{x\to\infty} \frac{1}{x} = 0$

This matches our intuition: dividing 1 by a really big number (∞) gives a really small number (0).

Just as before, we'd like to have tools to compute limits at infinity more easily.

FACTS:

1. The substitution principle still works for $\lim_{x\to\infty}$
2. The division by zero rule still works for $\lim_{x\to\infty}$
3. The MLT still works for $\lim_{x\to\infty}$ except for one key difference — we need to replace property (II) by: (II') $\lim_{x\to\infty} \frac{1}{x} = 0$
4. The squeeze theorem still works for $\lim_{x\to\infty}$

In other words, most of our tools still work. Let's practice:

EXAMPLES: Compute the following limits.

1. $\lim_{x\to\infty} \frac{7}{x}$.

   We use the MLT: $\lim_{x\to\infty} \frac{7}{x} = 7\left(\lim_{x\to\infty} \frac{1}{x}\right) = 7\cdot 0 = 0$

2. $\lim_{x\to\infty} \frac{1}{x^3}$

   Same idea: $\lim_{x\to\infty} \frac{1}{x^3} = \lim_{x\to\infty} \left(\frac{1}{x}\right)^3 = \left(\lim_{x\to\infty} \frac{1}{x}\right)^3 = 0^3 = 0$.

3. $\lim_{x\to\infty} \frac{x+1}{x^2}$.

Same idea again: $\lim_{x\to\infty} \frac{x+1}{x^2} = \lim_{x\to\infty}\left(\frac{x}{x^2}+\frac{1}{x^2}\right) = \lim_{x\to\infty}\left(\frac{x}{x^2}\right) + \lim_{x\to\infty}\left(\frac{1}{x^2}\right)$

$= \lim_{x\to\infty}\frac{1}{x} + \left(\lim_{x\to\infty}\frac{1}{x}\right)^2 = 0 + 0^2 = 0.$

(4) $\lim_{x\to\infty} \frac{2x}{x^2+3}$

We'd like to do the same, but we can't split up the denominator. So, we use a common technique: look for the biggest term — it's probably influencing everything else — and divide everything by it. In this case x^2 is the biggest thing, so:

$$\lim_{x\to\infty}\frac{2x}{x^2+3}\cdot\left(\frac{1/x^2}{1/x^2}\right) = \lim_{x\to\infty}\frac{2x/x^2}{(x^2+3)/x^2} = \lim_{x\to\infty}\frac{2/x}{1+3/x^2} \overset{*}{=} \frac{\lim_{x\to\infty} 2/x}{\lim_{x\to\infty}(1+3/x^2)} =$$

$$= \frac{2\lim_{x\to\infty} 1/x}{\lim_{x\to\infty} 1 + 3\left(\lim_{x\to\infty} 1/x\right)^2} = \frac{2\cdot 0}{1+3\cdot 0^2} = \frac{0}{1} = 0.$$

* Notice we used MLT a few times, and the starred step is okay since we found out (a few steps later) that the denominator is not zero.

We also used the substitution theorem here — the original function $\frac{2x}{x^2+3}$ is defined for any x, but the modified function $\frac{2/x}{1+3/x^2}$ is not defined at $x=0$. But this is okay because $x\to\infty$, so it's away from 0.

(5) $\lim_{x\to\infty} \frac{2x+1}{x-6}$:

Same idea: the largest power of x is $x^1 = x$. Dividing it out gives —

$$\lim_{x\to\infty}\frac{2x+1}{x-6}\cdot\left(\frac{1/x}{1/x}\right) = \lim_{x\to\infty}\frac{2+1/x}{1-6/x} = \frac{\lim_{x\to\infty}(2)+\lim_{x\to\infty}(1/x)}{\lim_{x\to\infty}(1)+\lim_{x\to\infty}(-6/x)} = \frac{2+0}{1-0} = 2.$$

(6) $\lim_{x\to\infty} \frac{x^3+1}{x^2-1}$

Same idea: divide by x^3 —

$$\lim_{x\to\infty}\frac{x^3+1}{x^2-1} = \lim_{x\to\infty}\frac{1+1/x^3}{1/x-1/x^3} \overset{*}{=} \frac{\lim_{x\to\infty}(1+1/x^3)}{\lim_{x\to\infty}(1/x-1/x^3)} = \frac{1+0}{0-0} = \frac{1}{0}.$$

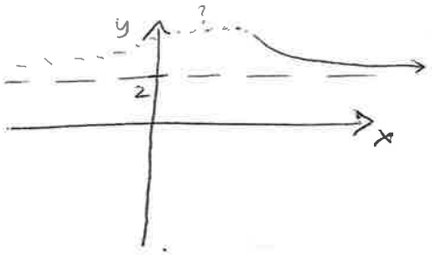
* Here, the starred step is not justified by MLT. But it's okay — we got a case where the numerator is nonzero and the denominator is zero. By the division-by-zero limit rule, $\boxed{\lim_{x\to\infty}\frac{x^3+1}{x^2-1} \text{ DNE}}$

(7) $\lim_{x\to\infty} \frac{\sin(x)}{x}$.

Remember $-1 \le \sin(x) \le 1$ for all x, so the numerator is "small" while the denominator is big. These inequalities suggest using the squeeze theorem. For $x>0$, we get,

$-\frac{1}{x} \le \frac{\sin(x)}{x} \le \frac{1}{x}$. Further, we know $\lim_{x\to\infty}\left(-\frac{1}{x}\right) = -\lim_{x\to\infty}\frac{1}{x} = -(0) = 0$ and $\lim_{x\to\infty}\left(\frac{1}{x}\right) = 0$. Since these are the same, the squeeze theorem tells us that $\boxed{\lim_{x\to\infty}\frac{\sin(x)}{x} = 0}$.

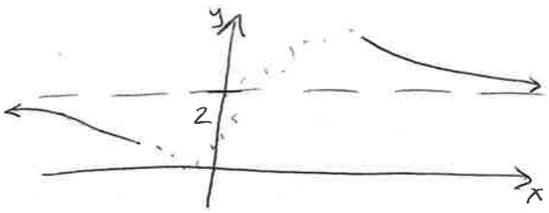
2 What do these results look like? For example, what does example (5) tell us about the graph of $y = \frac{2x+1}{x-6}$? Well, we know that as x gets really big, y approaches 2. So, on a graph, we would get something like:



I've drawn a dotted line at first, because we don't know what happens for "small" x, but once x is big enough, $y \to 2$.

In fact, we can also compute

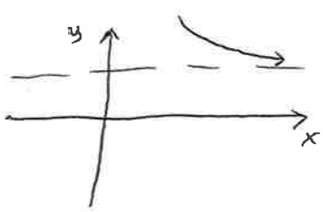
$$\lim_{x\to-\infty}\frac{2x+1}{x-6} = \lim_{x\to-\infty}\frac{2+1/x}{1-6/x} = \frac{2+0}{1+0} = 2$$ as well, so we can update our graph:

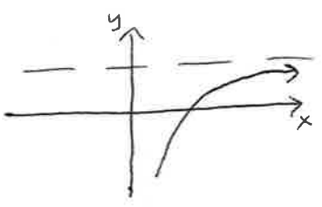


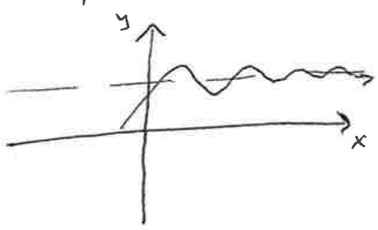
Again, we're not sure yet what happens in between.

In other words, we've seen that if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then $y = L$ is a horizontal asymptote for the graph of $y = f(x)$.

Horizontal asymptotes can approach from above, below, or even fluctuate:





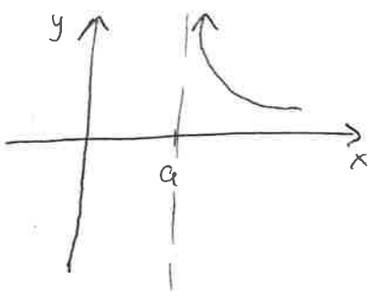


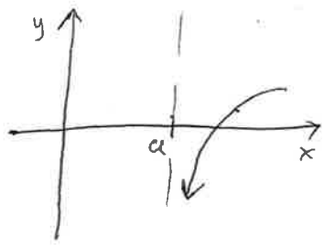
(so actually, we're not sure if our picture of $y = \frac{2x+1}{x-6}$ is that accurate).

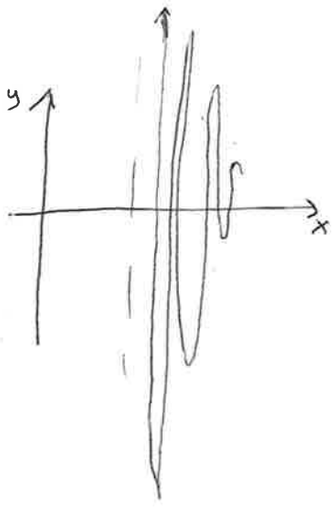
3 Finally, as long as we're discussing ~~vertical~~ horizontal asymptotes, we should discuss vertical ones as well. We start by improving our "division-by-zero" principle:

THEOREM: If $\lim_{x\to a} f(x) \neq 0$ and $\lim_{x\to a} g(x) = 0$ then $y = \frac{f(x)}{g(x)}$ has a vertical asymptote at $x = a$.

In this situation, we can ask (as before) if the graph goes "up" as it approaches the asymptote, goes "down", or fluctuates:







In other words, we look for each of the following cases:

$\lim_{x\to a^+} f(x) = \infty$ $\quad \lim_{x\to a^+} f(x) = -\infty$ $\quad \lim_{x\to a^-} f(x) = \infty$ $\quad \lim_{x\to a^-} f(x) = -\infty$,

EXAMPLE: Determine the vertical asymptotes of $f(x) = \frac{2x+1}{x-6}$.

The only chance of a vertical asymptote is when the denominator is zero: $x - 6 = 0$.

Solving gives $x = 6$. To check if it's an asymptote, take the limit:

$\lim_{x\to 6} \frac{2x+1}{x-6} = \frac{2(6)+1}{(6)-6} = \frac{13}{0}$ DNE so yes, it's an asymptote.

Which way does it go? Well, if $x > 6$, then $x - 6 > 0$ so $x-6$ is positive.

Also $2x > 12$ so $2x+1 > 13$ is also positive. So

$\lim_{x\to 6^+} \frac{2x+1}{x-6} = +\infty$

Similarly, if $x < 6$, then $x - 6 < 0$ is negative. We're thinking about $x \to 6$, so when x is close to 6, it's at least bigger than 5, i.e. $x > 5$, so $2x > 10$ and $2x+1 > 11$ is positive. So $f(x) = \frac{\text{positive}}{\text{negative}} = \text{negative}$, which means $\lim_{x\to 6^-} \frac{2x+1}{x-6} = -\infty$. Putting all our facts together (horizontal + vertical) gives a clearer picture of

$$y = \frac{2x+1}{x-6}$$

