§1.6

Main Concepts

1. Continuity
2. Compositions and substitutions
3. Miscellanea
4. Intermediate Value Theorem

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1 We have noticed at this point that to compute a limit, we're often just plugging in $x=a$ into our function $f(x)$. Actually, this happens precisely when $f(x)$ is continuous.

DEFINITION A function $f(x)$ is continuous at $x=a$ if $\lim_{x\to a} f(x) = f(a)$.

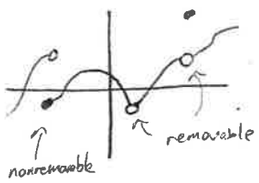
We've seen plenty of examples:

RECALL
- Rational functions are continuous on their domains.
- Trig functions ($\sin, \cos, \tan, \sec, \csc, \cot$) are also continuous on their domains.

EXAMPLE: Compute $\lim_{x\to 7} \frac{x^2+3}{x-3}$

This is a rational function, so $\lim_{x\to 7} f(x) = f(7) = \frac{7^2+3}{7-3} = \frac{49+3}{4} = \frac{52}{4} = 13$.

On the other hand, if $f(x)$ is not continuous at $x=a$, it has a discontinuity. There are two ways this can happen: if $\lim_{x\to a} f(x)$ does not exist, then it can't equal $f(a)$: this is a nonremovable discontinuity. If, however, $\lim_{x\to a} f(x)$ exists, but is different from $f(a)$, this is a removable discontinuity.



EXAMPLES: For each of the following, classify $x=0$ as removable or nonremovable.

① $f(x) = \frac{x^2+1}{x}$ Here, $\lim_{x\to 0} \frac{x^2+1}{x} = \frac{0^2+1}{0} = \frac{1}{0}$ DNE. So the discontinuity is nonremovable.

② $f(x) = \frac{\sin(x)}{x}$ Here, $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, so the discontinuity is removable.

③ $f(x) = \frac{|x|}{x}$ Here, $\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$ and

$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} (-1) = -1$. So $\lim_{x\to 0} f(x)$ DNE and $x=0$ is a nonremovable discontinuity

We call example ② "removable because we can "remove" the discontinuity by redefining $f(a)$. In example ②, we see now that $g(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x\neq 0 \\ 1 & \text{if } x=0 \end{cases}$ is continuous: we've removed the discontinuity.

However, this doesn't help us with examples like [illegible]. We need another theorem:

THEOREM If $f(x)$ and $g(x)$ are continuous, then so is $(f \circ g)(x) = f(g(x))$.

Let's see how to use this:

EXAMPLE: Compute $\lim_{x \to 2} \sin(x^2+1)$.

This is a composition. That is, if $f(x) = \sin(x)$ and $g(x) = x^2+1$, then $(f \circ g)(x) = f(g(x)) = \sin(x^2+1)$. We know $f(x)$ is continuous since it's a trig function, and we know $g(x)$ is continuous since it's a polynomial (rational function). So, by the above theorem, $(f \circ g)(x)$ is continuous. So:

$$\lim_{x \to 2} \sin(x^2+1) = \lim_{x \to 2} (f \circ g)(x) = (f \circ g)(2) = \sin((2)^2+1) = \sin(5).$$

There's another way to understand these compositions, and that's by changing variables. We know how to take $\lim_{x \to a} \sin(x)$, so in the above example, we wish we had x instead of x^2+1. So, let's make a new variable, y, to do just that: $y = x^2+1$. Then $\sin(x^2+1) = \sin(y)$. To take the limit, we also need to know where our new variable is going. In the example,

$x \to 2$, so $y = x^2+1 \longrightarrow (2)^2+1 = 4+1 = 5$. In other words, $\lim_{x \to 2} y = \lim_{x \to 2} (x^2+1) = 2^2+1 = 5$.

So, we can now redo this problem: $\lim_{x \to 2} \sin(x^2+1) = \lim_{y \to 5} \sin(y) = \sin(5)$.

Note that we got the same result.

EXAMPLE: Compute $\lim_{x \to 0} \frac{\sin(2x)}{x}$ using both composition approaches.

① For the first approach, we want to write $\frac{\sin(2x)}{x}$ in the form $f(g(x))$ for some f, g, continuous. We see $2x$ inside the parentheses, so let's try $g(x) = 2x$. So, we want to write $\frac{\sin(2x)}{x} = f(2x)$. In other words, we want everything in terms of $2x$:

$\frac{\sin(2x)}{x} \cdot \frac{2}{2} = \frac{2\sin(2x)}{2x}$. So, if $f(x) = \frac{2\sin(x)}{x}$, then $f(g(x)) = \frac{2\sin(2x)}{(2x)} = \frac{\sin(2x)}{x}$ as desired. But there's a problem: $f(x)$ is not continuous at $x=0$! It's okay though; the discontinuity is removable since $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{2\sin(x)}{x} = 2\lim_{x \to 0} \frac{\sin(x)}{x} = 2(1) = 2$.

So really, we should define $f(x) = \begin{cases} \frac{2\sin(x)}{x} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$. Now f is continuous (we've removed the discontinuity) and so

$$\lim_{x \to 0} \frac{\sin(2x)}{x} = \lim_{x \to 0} f(g(x)) = f(g(0)) = f(2 \cdot 0) = f(0) = 2$$

② Our second approach starts similarly: we want to simplify the $2x$ inside $\sin(2x)$. So, we choose $y = 2x$. Then $\lim_{x\to 0} y = \lim_{x\to 0}(2x) = 2(0) = 0$, so $y \to 0$, too. Then

$$\lim_{x\to 0}\frac{\sin(2x)}{x} = \lim_{y\to 0}\frac{\sin(y)}{(y/2)}\star = \lim_{y\to 0}\frac{2\sin(y)}{y} = 2\lim_{y\to 0}\frac{\sin(y)}{y} = 2(1) = 2.$$

(★ Here, we solved for x: $y = 2x$, so $x = y/2$)

Notice we got the same result, but the 2nd approach was much simpler this time. Finally, there is a 3rd approach to compositions: if $f(x)$ and $g(x)$ are continuous, then

$$\lim_{x\to a} f(g(x)) = f(g(a)) = f\left(\lim_{x\to a} g(x)\right) \quad \text{since } \lim_{x\to a} g(x) = g(a).$$

In other words, we can "bring the limit inside" of continuous functions.

EXAMPLE: Compute $\lim_{x\to 0}\cos(x^3+2x+\pi)$ using each approach

① Take $f(x) = \cos(x)$ and $g(x) = x^3+2x+\pi$. f and g are continuous (trig + rational), so $f(g(x)) = \cos(x^3+2x+\pi)$ is continuous, too. So:

$$\lim_{x\to 0}\cos(x^3+2x+\pi) = \lim_{x\to 0} f(g(x)) = f(g(0)) = \cos(0^3+2\cdot 0+\pi) = \cos(\pi) = -1.$$

② Let $y = x^3+2x+\pi$. Then $\lim_{x\to 0} y = \lim_{x\to 0}(x^3+2x+\pi) = (0)^3+2(0)+\pi = \pi$ so $y \to \pi$.

Thus, $\lim_{x\to 0}\cos(x^3+2x+\pi) = \lim_{y\to\pi}\cos(y) = \cos(\pi) = -1$.

③ Directly: $\lim_{x\to 0}\cos(x^3+2x+\pi) = \cos\left(\lim_{x\to 0}(x^3+2x+\pi)\right) = \cos\left((0)^3+2(0)+\pi\right) = \cos(\pi) = -1$.

3 Between rational, trig, and composite functions, we see that most functions we'll run into are continuous. Here's a few more things to know:

FACTS Suppose $f(x)$ and $g(x)$ are continuous.

① $f(x)+g(x)$ is continuous

② $f(x)\cdot g(x)$ is continuous

③ $\frac{f(x)}{g(x)}$ is continuous when $g(x)\neq 0$.

④ $h(x) = |x|$ is continuous

⑤ $h(x) = \sqrt[n]{x}$ is continuous on its domain.

EXAMPLE: Show that $f(x) = (\sin(x+1))^{2/3}$ is continuous.

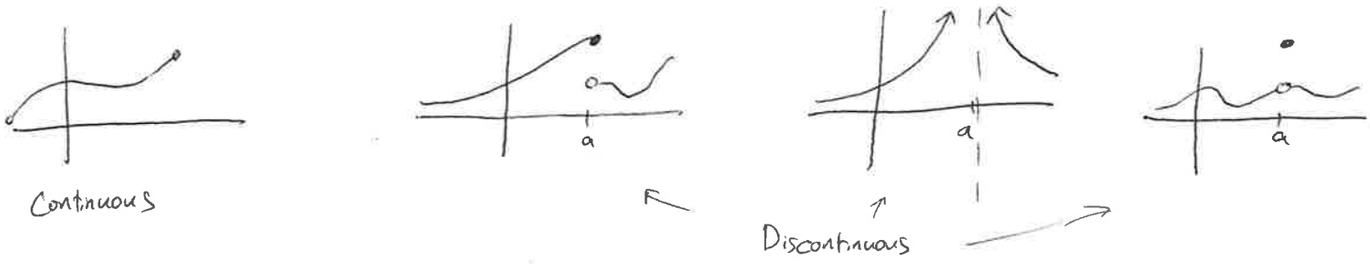
Notice $g(x) = \sin(x)$ and $h(x) = x+1$ and $i(x) = x^2$ and $j(x) = \sqrt[3]{x}$ are continuous.

So, $j(i(g(h(x))))$ is continuous. But

$$j(i(g(h(x)))) = j(i(g(x+1))) = j(i(\sin(x+1))) = j((\sin(x+1))^2) = \sqrt[3]{(\sin(x+1))^2} = (\sin(x+1))^{2/3}.$$

So we're done.

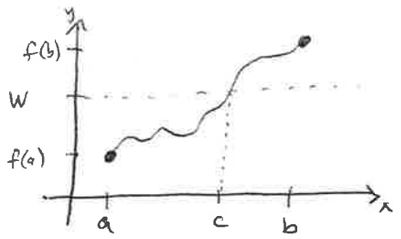
4 Finally, let's talk about what continuity means graphically. As x approaches a, $f(x)$ approaches $f(a)$. So, in other words, as I move my pen on the page, I don't "jump" at a, for any a.



Continuous

Discontinuous

This gives a useful result. If I draw the graph from $x=a$ to $x=b$ and y goes from $y=f(a)$ to $y=f(b)$, I must pass through every number between $f(a)$ & $f(b)$ at least once.



In other words:

THEOREM (Intermediate Value Theorem or IVT): If $f(x)$ is continuous and W is between $f(a)$ and $f(b)$, then $f(c)=W$ for some $a<c<b$.

EXAMPLES:

① Show that $x^2+x=5$ for some $1<x<2$.

Notice $f(x)=x^2+x$ is continuous and $f(1)=1^2+1=2$ and $f(2)=2^2+2=4+2=6$.

Since $2<5<6$, $f(x)=5$ for some $1<x<2$, i.e. $f(x)=x^2+x=5$, by the intermediate value theorem.

② Show that $x^3-2x=3$ for some x.

This time, I haven't given a a & b, so we should find them ourselves. Just guessing $a=0$ gives $f(0)=(0)^3-2(0)=0$. So, we want $f(b)>3$ so that $f(a)<3<f(b)$. Let's try $b=1$: $f(1)=1^3-2(1)=1-2=-1$. That didn't work, so try $b=2$: $f(2)=(2)^3-2(2)=8-4=4$. This worked: $f(0)<3<f(2)$ and f is continuous, so $f(x)=3$ for some $0<x<2$.

③ Show $\cos(x)=x$ for some x.

USEFUL TECHNIQUE Start by moving everything to one side: $f(x)=\cos(x)-x$. This is continuous, so let's plug in values. $f(0)=\cos(0)-(0)=1$. $f(\pi)=\cos(\pi)-\pi=-1-\pi$. Notice $f(0)$ is positive and $f(\pi)$ is negative, so $f(0)>0>f(\pi)$. By IVT, $f(x)=0$ for some $0<x<\pi$. I.e. $\cos(x)-x=0$, so $\cos(x)=x$ for this x.