§2.2

Main Concepts

1. Derivatives
2. Graphing
3. Differentiability
4. Notation

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1 The concept from the last section is important enough to have its own name:

DEFINITION: Let $f(x)$ be a function. Then the derivative of f is the function

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \qquad \text{[pronounced "f prime"]}.$$

EXAMPLES: For each function $f(x)$, compute $f'(0)$ and $f'(1)$.

① $f(x) = 2x+3$

Directly, $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{[2(h)+3]-[2(0)+3]}{h} = \lim_{h\to 0} \frac{2h+3-3}{h} =$

$= \lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} 2 = \boxed{2}$.

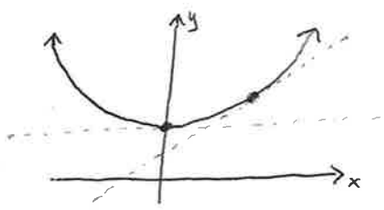
and $f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{[2(1+h)+3]-[2(1)+3]}{h} = \lim_{h\to 0} \frac{2h+5-5}{h} = \boxed{2}$

This makes sense: a straight line has (rate of change) = (slope) = (constant) — in this case, $= 2$.

② $f(x) = x^2+3$

$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{[(h)^2+3]-[(0)^2+3]}{h} =$

$= \lim_{h\to 0} \frac{h^2+3-3}{h} = \lim_{h\to 0} \frac{h^2}{h} = \lim_{h\to 0} (h) = \boxed{0}$



and $f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{[(1+h)^2+3]-[(1)^2+3]}{h} =$

$= \lim_{h\to 0} \frac{h^2+2h+4-4}{h} = \lim_{h\to 0} (h+2) = 0+2 = \boxed{2}$

These results also make sense: when $x=0$, the graph is "flat" (slope $=0$) and when $x=2$, it's increasing (slope $=2$).

③ $f(x) = \frac{1}{x+1}$

$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{\left[\frac{1}{(h)+1}\right]-\left[\frac{1}{(0)+1}\right]}{h} = \lim_{h\to 0} \frac{\frac{1}{h+1}-1}{h} = \lim_{h\to 0} \frac{\frac{1}{h+1}-\frac{h+1}{h+1}}{h} =$

$= \lim_{h\to 0} \frac{\frac{-h}{h+1}}{h} = \lim_{h\to 0} -\frac{h}{h(h+1)} = -\lim_{h\to 0} \frac{1}{h+1} = -\frac{1}{(0)+1} = \boxed{-1}$

and $f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{\left[\frac{1}{(1+h)+1}\right]-\left[\frac{1}{(1)+1}\right]}{h} = \lim_{h\to 0} \frac{\frac{1}{h+2}-\frac{1}{2}}{h} = \lim_{h\to 0} \frac{\frac{2}{2(h+2)}-\frac{h+2}{2(h+2)}}{h} =$

$= \lim_{h\to 0} \frac{\left(\frac{-h}{2(h+2)}\right)}{h} = \lim_{h\to 0} -\frac{h}{2h(h+2)} = -\lim_{h\to 0} \frac{1}{2(h+2)} = -\frac{1}{2(0+2)} = \boxed{-1/4}$

We can also do this in general, rather than for a specific x.

EXAMPLE: For $f(x) = x^2$, find $f'(x)$.

Directly, $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - (x)^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2 - x^2}{h} =$

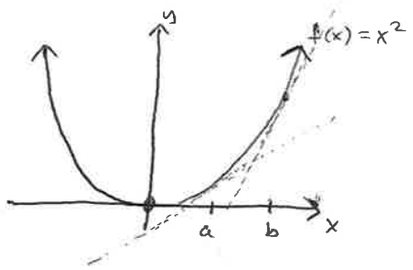
$$= \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} (2x+h) = 2x+(0) = 2x$$

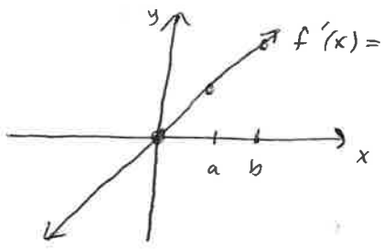
Now, we can find all values of the derivative: $f'(0) = 2(0) = 0$

$f'(1) = 2(1) = 2$

$f'(-3) = 2(-3) = -6$, etc...

2. I've hinted at this already in the examples, but a powerful way to study the derivative is by graphing. If $y = f(x)$, then $f'(x)$ is the slope at the point $(x, f(x))$ which we can determine by zooming in. Let's start by re-interpreting this last example:



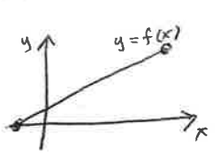


What are some salient features?

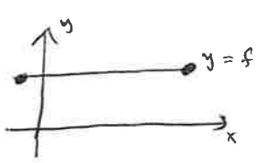
- When $f(x)$ is "flat" (minimum, valley, peak, maximum, etc.) $f'(x) = 0$
- When $f(x)$ is increasing, $f'(x) > 0$ and when $f(x)$ is decreasing, $f'(x) < 0$
- If $f(x)$ is increasing faster at b than at a, then $f'(b) > f'(a)$ (steep slope vs. gentle slope).

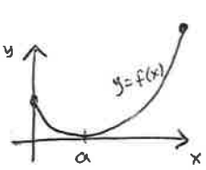
We can use these facts to reverse the problem: given the graph of $y = f(x)$, draw a graph of $y = f'(x)$.

EXAMPLES: Given the graph of $y = f(x)$, draw the graph of $y = f'(x)$.

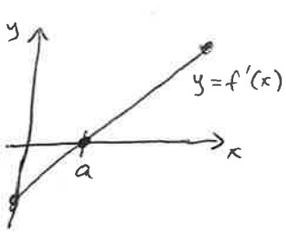
① 

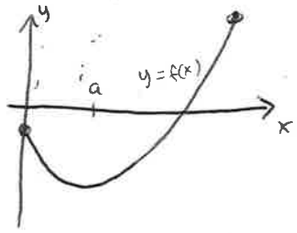
Here, the slope is positive and constant. So, $f'(x)$ is positive and constant



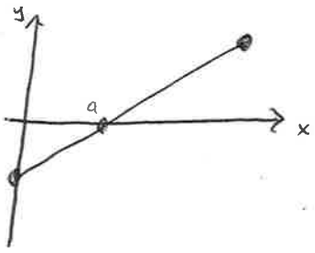
② 

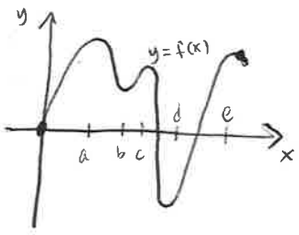
The function is flat at a single point, labelled "a". For $x < a$, $f(x)$ is decreasing, so $f'(x) < 0$. For $x > a$, $f(x)$ is increasing, so $f'(x) > 0$. Finally, we can see that the slope is always increasing (steep negative → gentle negative → flat → gentle increase → steep increase) so $f'(x)$ is increasing.



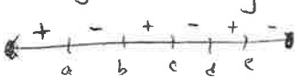
③ 

The exact same analysis applies to this graph as in ②. So, $f'(x)$ will look the same. In particular, it doesn't matter if $f(x)<0$ since we're only looking at slopes.

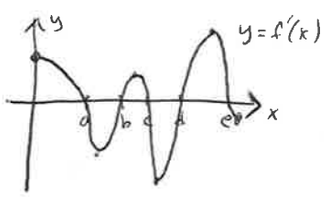


④ 

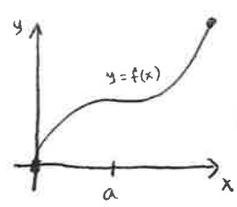
Immediately, we label the five points (a, b, c, d, e) where $f'(x)=0$. Now, we ask if $f(x)$ is increasing or decreasing in between. This diagram



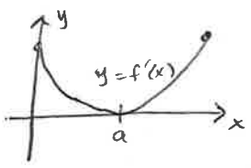
captures the information.



Finally, (qualitatively) the steepest increase is (probably) between d & e, followed by the initial slope at $x=0$, trailed by $b \to c$. The steepest decrease is $c \to d$, followed by $a \to b$.

⑤ 

At $x=a$, the function does flatten out. However, this time, $f(x)$ is increasing for $x<a$ and $x>a$. So $f'(x)>0$ for all $x \neq a$. It's also steepest at the two endpoints.



3 Now, a technical note. We're computing $f'(x)$ by taking a limit, and we know that sometimes limits do not exist. When $f'(x)$ exists, we say $f(x)$ is differentiable. Let's see an example that isn't differentiable:

EXAMPLES: Let $f(x)=|x|$. For each value of x, either compute $f'(x)$ or show that $f(x)$ is not differentiable for that x.

① $x=1$. Here, $f'(x)=f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{|h+1|-|1|}{h}\overset{\star}{=}\lim_{h\to 0}\frac{(h+1)-(1)}{h}=\lim_{h\to 0}\frac{h}{h}=\lim_{h\to 0}1=1$

Notice that in step ★, we use the substition principle: if h is close to zero, then $1+h$ is positive, so $|1+h|=1+h$.

② $x=-1$. Same idea: $f'(x)=f'(-1)=\lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}=\lim_{h\to 0}\frac{|-1+h|-|-1|}{h}\overset{\star}{=}\lim_{h\to 0}\frac{-(-1+h)-(1)}{h}=\lim_{h\to 0}\frac{-h+1-1}{h}=$

$$=\lim_{h\to 0}\frac{-h}{h}=\lim_{h\to 0}(-1)=-1.$$

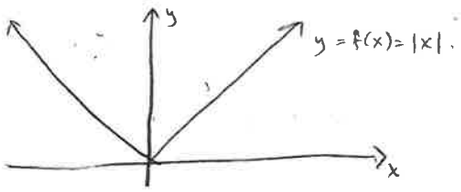
Again, in ★, we use that if $h \approx 0$, then $-1+h$ is negative, so $|-1+h|=-(-1+h)$.

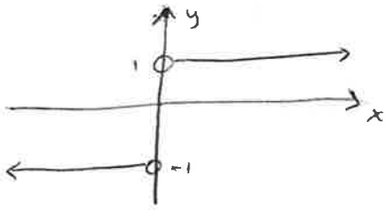
③ $x=0$. Start the same way: $f'(x)=f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{|h|-|0|}{h}=\lim_{h\to 0}\frac{|h|}{h}$ DNE.

We've seen this limit before and know it doesn't exist. The problem is that "★" doesn't work — if $h \approx 0$ then we don't know if h is positive or negative, so we don't know if $|h|=h$ or $|h|=-h$.

So $f(x)=|x|$ is not differentiable at $x=0$, but it is differentiable at $x=1$ and $x=-1$.

We can understand this conclusion graphically as well. For $x<0$, $|x|=-x$ is a straight line with slope -1. And for $x>0$, $|x|=x$ is a line with slope 1. But at

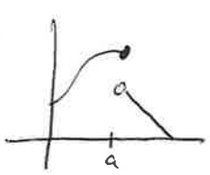




$x=0$ itself, there's no clear "slope." This is one form of non-differentiability — any graph with a sharp "corner" won't be differentiable. [Cautionary note: graphs can be misleading. $y = \frac{10x^2-10}{10x^2+1}$, graphed from $-5 \le x \le 5$, $-10 \le y \le 2$ might look like it has a sharp corner, but it goes away when you zoom in.]

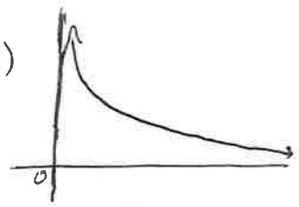
The other common forms of nondifferentiability are 1) a discontinuity, 2) an asymptote, and 3) an "infinite" slope. In graphs:

1)



Not differentiable at $x=a$

2)



Not differentiable at $x=0$

3)



Not differentiable at $x=0$ (too steep!).

Here's an algebraic example of each:

EXAMPLES: Show each function is not differentiable at $x=0$

① $f(x) = \begin{cases} x+1 & \text{if } x<0 \\ x-1 & \text{if } x \ge 0 \end{cases}$ We have $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)-(0-1)}{h} = \lim_{h\to 0} \frac{f(h)+1}{h}$.

We'll compute the left limit first: $\lim_{h\to 0^-} \frac{f(h)+1}{h} = \lim_{h\to 0^-} \frac{(h+1)+1}{h} = \lim_{h\to 0^-} \frac{h+2}{h} = \frac{2}{0}$ DNE.

② $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \ne 0 \\ 0 & \text{if } x=0 \end{cases}$ Now $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{(1/h)-(0)}{h} = \lim_{h\to 0} \frac{1}{h^2} = \frac{1}{0^2} = \frac{1}{0}$

DNE

③ $f(x) = \sqrt{x}$ Now $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{\sqrt{h}-\sqrt{0}}{h} = \lim_{h\to 0} \frac{\sqrt{h}}{h} = \lim_{h\to 0} \frac{1}{\sqrt{h}} = \frac{1}{\sqrt{0}} \ne \frac{1}{0}$

DNE

Finally, a brief note on notation. So far, I've been writing $f'(x)$ for the derivative. A second notation that is common comes from Leibniz: write slope $= \frac{y_2-y_1}{x_2-x_1} = \frac{\Delta y}{\Delta x}$, we take the limit and replace "Δ" with "d" to get derivative $= \frac{dy}{dx}$. So, if $y=x^2$, we could write $\frac{dy}{dx} = 2x$ (analog of $f(x)=x^2 \longrightarrow f'(x)=2x$). We also use this notation as a verb: $\frac{dy}{dx}$ means the derivative of y with respect to x. So, $\frac{d}{dx}$ is just the action — "take the derivative" So, we could also write $\frac{d}{dx}(x^2) = 2x$

("Take the derivative of x^2 and get $2x$").