## §2.3 Main Concepts

1. Lines and Multiples
2. Sum ~~Rule~~ and Power Rule
3. ~~Power Rule~~ Examples
4. Product and Quotient Rules
5. Summary

---

1. Now that we've established the definition of the derivative, we'd like to have rules (analogous to the main limit theorem) to handle more complicated derivatives. We'll start easy:

If we take $f(x)=mx+b$, then we know the graph of $y=f(x)$ is a straight line with slope $m$. So, we expect $f'(x)=m$ for all $x$, and we can confirm this—

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{[m(x+h)+b]-[mx+b]}{h}=\lim_{h\to 0}\frac{mx+mh+b-mx-b}{h}=\lim_{h\to 0}\frac{mh}{h}=m.$$

The algebra confirms this, but it's useful to remember that we can discover the answer by thinking geometrically (slope). Similarly, we can imagine that if we take a function and double it, the slope should also double. E.g. if two runners start at the 200-meter mark, and runner A is always at twice the distance as runner B, then we conclude that runner A is moving twice as fast. In summary, we're saying that if $k$ is a constant, then $\boxed{\frac{d}{dx}(kf(x))=k\frac{d}{dx}(f(x))}$.

We can also reach another conclusion. If $m=0$ above, then $f(x)=mx+b=b$ is a constant function. In this case, we've learned $\boxed{\frac{d}{dx}(b)=0}$.

2. Our next tool is to add derivatives together. Again, the geometry helps us: if $f(x)$ has slope $m$ and $g(x)$ has slope $n$, then $f(x)+g(x)$ should have slope $m+n$, since increasing $x$ by a little increases $f(x)$ and $g(x)$ independently. So: $\boxed{\frac{d}{dx}(f(x)+g(x))=\frac{d}{dx}(f(x))+\frac{d}{dx}(g(x))}$

We know now how to put pieces together (adding two functions, multiplying one by a constant), but we don't have many pieces. Let's create some simple ones. First, $\frac{d}{dx}(x)=1$ since $f(x)=x$ is a line. Second,

$$\frac{d}{dx}(x^2)=\lim_{h\to 0}\frac{(x+h)^2-(x)^2}{h}=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}=\lim_{h\to 0}(2x+h)=2x$$

as we showed before. With some work (lots of FOILing), we can do $x^3$:

$$\frac{d}{dx}(x^3)=\lim_{h\to 0}\frac{(x+h)^3-(x)^3}{h}=\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}=\lim_{h\to 0}(3x^2+3xh+h^2)=3x^2.$$

Finally, $x^4$:

$$\frac{d}{dx}(x^4)=\lim_{h\to 0}\frac{(x+h)^4-(x)^4}{h}=\lim_{h\to 0}\frac{x^4+4x^3h+6x^2h^2+4xh^3+h^4-x^4}{h}=\lim_{h\to 0}(4x^3+6x^2h+4xh^2+h^3)=4x^3.$$

We may, at this point, suspect a pattern. And indeed, it's true, and gives

the "Power Rule" — $\boxed{\frac{d}{dx}(x^n) = nx^{n-1}}$.

3 We have lots of tools — let's use them!

— EXAMPLES: For each function $f(x)$, find $f'(x)$.

① $f(x) = x^{13}$

By the power rule, $f'(x) = 13x^{12}$

② $f(x) = 6x^5$

We have $f'(x) = \frac{d}{dx}(6x^5) = 6 \cdot \frac{d}{dx}(x^5) = 6 \cdot (5x^4) = 30x^4$.

③ $f(x) = x^2 + x^3$

We have $f'(x) = \frac{d}{dx}(x^2+x^3) = \frac{d}{dx}(x^2) + \frac{d}{dx}(x^3) = (2x) + (3x^2) = 2x + 3x^2$.

④ $f(x) = 2x^6 - x^7 + x - 10$

We have $f'(x) = \frac{d}{dx}(2x^6 - x^7 + x - 10) = \frac{d}{dx}(2x^6) + \frac{d}{dx}(-x^7) + \frac{d}{dx}(x) + \frac{d}{dx}(-10) =$

$= 2 \cdot \frac{d}{dx}(x^6) - \frac{d}{dx}(x^7) + \frac{d}{dx}(x) + \frac{d}{dx}(-10) =$

$= 2 \cdot (6x^5) - (7x^6) + (1) + (0) = 12x^5 - 7x^6 + 1.$

⑤ $f(x) = (x+2)^2$

We don't have a rule for function composition $[g(h(x))]$ yet, so we have to expand first:

$f(x) = x^2 + 4x + 4$ Then, $f'(x) = \frac{d}{dx}(x^2+4x+4) = \frac{d}{dx}(x^2) + \frac{d}{dx}(4x) + \frac{d}{dx}(4) =$

$= 2x + 4$

⑥ $f(x) = x^4 + 2x^3 + 3x^2 + 4x + 5$

At this point, we get it, and we'll stop writing the intermediate steps (sums & constants).

$f'(x) = 4x^3 + 2(3x^2) + 3(2x) + 4(1) + (0) = 4x^3 + 6x^2 + 6x + 4.$

4 So far so good. Next, we ask about the derivative of a product. We would hope that we multiply the derivatives, but it's not so! In fact, this is not so surprising; if we have a ~~square~~ rectangular plot of land, and one side starts getting longer, the area increases even if the other side stays constant. In other words, it's possible that $f'(x) > 0$ and $g'(x) = 0$ but $(f(x)g(x))' \neq 0$. In fact, we have

$\boxed{\text{PRODUCT RULE: } \frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).}$

In words: "to take the derivative of a product, differentiate the first and leave the second alone, then differentiate the second and leave the first alone, and add them."

**EXAMPLES**: Find $f'(x)$ for each of the following.

① $f(x) = (x^3+2x+1)(x^2-4)$

Note: we could solve this by first expanding and then differentiating. Instead, we'll use the product rule:

$$f'(x) = (3x^2+2+0)(x^2-4) + (x^3+2x+1)(2x-0)$$
$$= (3x^2+2)(x^2-4) + (x^3+2x+1)(2x).$$

[We could expand this to simplify a little, but if we were going to do that, we should have just expanded first!]

② $f(x) = (x^2+1)^2$

We can write this as $f(x) = (x^2+1)\cdot(x^2+1)$ which makes the product rule cleaner:

$$f'(x) = (2x+0)(x^2+1) + (x^2+1)(2x+0) = 2\big(2x(x^2+1)\big) = 4x(x^2+1).$$

**EXERCISE** Do the above exercises by expanding first.

Similarly, we can take the derivative of fractions, but it's not simply the ratio of the derivatives. Instead, we have another rule:

**QUOTIENT RULE**: $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

(Again, in words: The derivative of "top over bottom" is top' times bottom minus top times bottom' all over bottom-squared.)

**EXAMPLES**: Find $f'(x)$ for each of the following.

① $f(x) = \dfrac{1}{x}$.

Using the quotient rule, $f'(x) = \dfrac{(1)'(x) - (1)(x)'}{(x)^2} = \dfrac{(0)(x) - (1)(1)}{(x)^2} = \dfrac{-1}{x^2}$.

Notice the power rule also works here: $f(x) = x^{-1}$ so $f'(x) = (-1)x^{-1-1} = -1\cdot x^{-2} = \dfrac{-1}{x^2}$.

② $f(x) = \dfrac{1}{x^2}$

Using the quotient rule, $f'(x) = \dfrac{(1)'(x^2) - (1)(x^2)'}{(x^2)^2} = \dfrac{(0)(x^2) - (1)(2x)}{x^4} = \dfrac{-2x}{x^4} = -\dfrac{2}{x^3}$.

Again, notice the power rule would have worked: $f(x) = x^{-2}$ so $f'(x) = (-2)x^{-3} = \dfrac{-2}{x^3}$.

③ $f(x) = \dfrac{x^2-2x-6}{x+1}$

The power rule doesn't help this time! We have to do a quotient —

$$f'(x) = \frac{(x^2-2x-6)'(x+1) - (x^2-2x-6)(x+1)'}{(x+1)^2} = \frac{(2x-2)(x+1) - (x^2-2x-6)(1+0)}{(x+1)^2} =$$
$$= \frac{2x^2 - 2 - x^2 + 2x + 6}{(x+1)^2} = \frac{x^2+2x+4}{(x+1)^2}$$

5 We've developed a lot of rules, so I'll summarize them here in one place:

- CONSTANT RULE: $\frac{d}{dx}(k) = 0$ (derivative of a constant is zero)

- SUM RULE: $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$.

- COEFFICIENT RULE: $\frac{d}{dx}(k \cdot f(x)) = k \cdot \frac{d}{dx}(f(x))$

- POWER RULE: $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$

- PRODUCT RULE: $\frac{d}{dx}(f(x) \cdot g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + f(x) \cdot \frac{d}{dx}(g(x))$.

- QUOTIENT RULE: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - f(x) \cdot \frac{d}{dx}(g(x))}{[g(x)]^2}$