§2.4

Main Concepts

1. Sine and Cosine
2. Other trig functions

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1. With the sum, coefficient, and power rules, we can handle polynomials well. But what about the derivatives of trig functions? We have to go back to the definition for $f(x) = \sin(x)$ —

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} =$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} = \quad [\text{Trig identity : sin of sum}]$$

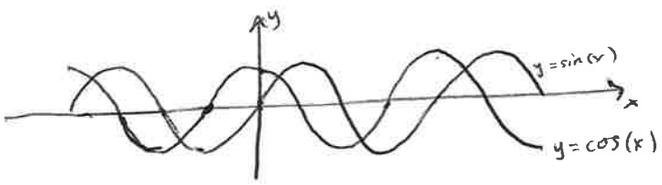
$$= \lim_{h\to 0}\left[\frac{\sin(x)\cos(h)-\sin(x)}{h} + \frac{\cos(x)\sin(h)}{h}\right] = -\sin(x)\lim_{h\to 0}\left(\frac{1-\cos(h)}{h}\right) + \cos(x)\lim_{h\to 0}\left(\frac{\sin(h)}{h}\right) =$$

$$= -\sin(x)\cdot(0) + \cos(x)\cdot(1) = \quad [\text{Memorized trig limits}]$$

$$= \cos(x).$$

In other words, $\boxed{\frac{d}{dx}(\sin(x)) = \cos(x)}$. A similar calculation shows that $\boxed{\frac{d}{dx}(\cos(x)) = -\sin(x)}$.

We can check if we believe this by graphing: Notice that, as expected, when $\sin(x)$ is



increasing, $\cos(x) > 0$; when $\sin(x)$ is max/min, $\cos(x) = 0$; and when $\sin(x)$ is decreasing, $\cos(x) < 0$.

A similar analysis helps us believe that $\frac{d}{dx}(\cos(x)) = -\sin(x)$.

2. With the quotient rule, we can now work out the remaining trig functions. ~~For example~~

EXAMPLES: Find $f'(x)$

① $f(x) = \tan(x)$.

We can rewrite $\tan(x) = \frac{\sin(x)}{\cos(x)}$, so by the quotient rule,

$$f'(x) = \frac{\frac{d}{dx}(\sin(x))\cos(x) - \sin(x)\cdot\frac{d}{dx}(\cos(x))}{[\cos(x)]^2} = \frac{(\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))}{\cos^2(x)} =$$

$$= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x).$$

② $f(x) = \sec(x)$.

Again, rewriting this as $f(x) = \frac{1}{\cos(x)}$ gives

$$f'(x) = \frac{(1)'(\cos(x)) - (1)(\cos(x))'}{[(\cos(x))]^2} = \frac{(0)(\cos(x)) - (1)(-\sin(x))}{\cos^2(x)} = \frac{\sin(x)}{\cos^2(x)} = \sec(x)\tan(x).$$

Similarly, we can get all of the remaining trig functions. In summary:

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

$$\frac{d}{dx}(\tan(x)) = \sec^2(x)$$

$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$$

$$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$$

$$\frac{d}{dx}(\cot(x)) = -\csc^2(x).$$

We can combine these to do more complicated functions.

EXAMPLES: Find $f'(x)$.

① $f(x) = x\sin(x)$.

By the product rule, $f'(x) = (x)'(\sin(x)) + (x)(\sin(x))' = (1)(\sin(x)) + (x)(\cos(x)) = \sin(x) + x\cos(x)$.

② $f(x) = \sin(x)\cos(x)$

By the product rule, $f'(x) = (\sin(x))'(\cos(x)) + (\sin(x))(\cos(x))' = (\cos(x))(\cos(x)) + (\sin(x))(-\sin(x)) =$

$$= \cos^2(x) - \sin^2(x).$$

③ $f(x) = \frac{\sin(x)}{x}$

By the quotient rule, $f'(x) = \frac{(\sin(x))'(x) - (\sin(x))(x)'}{(x)^2} = \frac{(\cos(x))(x) - (\sin(x))(1)}{x^2} = \frac{x\cos(x) - \sin(x)}{x^2}$