§2.5

## Main Concepts

1. Chain Rule
2. Change-of-variables form

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1 How would we take the derivative of $(x^2+3x+1)^2$? From §2.3, we know we can either expand this first, or use the product rule. What about $(x^2+3x+1)^4$? We could write

$(x^2+3x+1)^4 = (x^2+3x+1)^2 \cdot (x^2+3x+1)^2$ and use the product rule, but then each piece is like the first problem above, and so requires its own work. We could also expand it, but we'll get a big polynomial $(x^8 + \cdots)$. It would be nice to somehow use the fact that we know the derivatives of $x^2+3x+1$ and $x^4$ and put those results together. If the exponents is larger [$(x^2+3x+1)^{100}$, maybe?] it won't just be nice — it's necessary.

The useful result here is the Chain rule:

**CHAIN RULE**: $[f(g(x))]' = f'(g(x)) \cdot g'(x)$.

In words: "To take the derivative of a composition, differentiate the outside function normally, then multiply by the derivative of the inside function."

**EXAMPLES**: Find $f'(x)$,

① $f(x) = (x^2+3x+1)^2$

We want to use the power rule to bring the exponent down, and the chain rule lets us:

$$f'(x) = 2(x^2+3x+1)^1 \cdot (2x+3+0) = 2(2x+3)(x^2+3x+1).$$

② $f(x) = (x^2+3x+1)^4$

Same idea: $f'(x) = 4(x^2+3x+1) \cdot (2x+3+0) = 4(2x+3)(x^2+3x+1)^3$

③ $f(x) = (x^2+3x+1)^{100}$

This is no harder, now: $f'(x) = 100(x^2+3x+1)^{99} \cdot (2x+3) = 100(2x+3)(x^2+3x+1)^{99}$.

④ $f(x) = \sin(2x)$.

Again, this just looks like differentiating sin, but we have an inside function:

$$f'(x) = \cos(2x) \cdot \frac{d}{dx}(2x) = \cos(2x) \cdot 2 = 2\cos(2x).$$

⑤ $f(x) = \cos(x^2+2x)$

Same idea: $f'(x) = -\sin(x^2+2x) \cdot \frac{d}{dx}(x^2+2x) =$ ~~$\sin(x)(2x+2) - (2x+2)\sin$~~

$$= -\sin(x^2+2x)(2x+2) = -(2x+2)\sin(x^2+2x).$$

⑥ $f(x) = (\sin(x)\cos(x) + 6)^{10}$

We have $f'(x) = 10(\sin(x)\cos(x)+6)^9 \cdot \frac{d}{dx}[\sin(x)\cos(x)+6] =$ {Chain Rule}

$= 10(\sin(x)\cos(x)+6)^9\left[\frac{d}{dx}(\sin(x))\cos(x) + \sin(x)\frac{d}{dx}(\cos(x)) + 0\right] =$ {Product rule}

$= 10(\sin(x)\cos(x)+6)^9[\cos^2(x) - \sin^2(x)]$

Notice we sometimes need to combine rules in various steps.

⑦ $f(x) = (\sin(x^2+1) - 2x)^8$

Same idea: $f'(x) = 8(\sin(x^2+1)-2x)^7 \cdot \frac{d}{dx}[\sin(x^2+1) - 2x] =$ {Chain Rule}

$= 8(\sin(x^2+1)-2x)^7 \cdot \left[\cos(x^2+1)\cdot\frac{d}{dx}[x^2+1] - 2\right] =$ {Chain Rule}

$= 8(\sin(x^2+1)-2x)^7(2x\cdot\cos(x^2+1) - 2)$.

Here, we even had nested Chain Rules

⑧ $f(x) = \cos\left(\frac{x+1}{x-1}\right)$

$f'(x) = -\sin\left(\frac{x+1}{x-1}\right)\cdot\frac{d}{dx}\left[\frac{x+1}{x-1}\right] =$ {Chain Rule}

$= -\sin\left(\frac{x+1}{x-1}\right)\cdot\frac{\frac{d}{dx}(x+1)(x-1) - (x+1)\frac{d}{dx}(x-1)}{(x-1)^2} =$ {Quotient Rule}

$= -\sin\left(\frac{x+1}{x-1}\right)\cdot\frac{(x-1)-(x+1)}{(x-1)^2} = -\sin\left(\frac{x+1}{x-1}\right)\cdot\frac{-2}{(x-1)^2} =$

$= \frac{2}{(x-1)^2}\cdot\sin\left(\frac{x+1}{x-1}\right)$

With all of our techniques, we can compute fairly complex derivatives!

2 Whenever we have a composition $f(g(x)) = (f\circ g)(x)$, we have another way to look at it. Writing $y = f(g(x))$ and $u = g(x)$, we can think of this as a coordinate substition: $y = f(u)$. Then, we're looking for $\frac{d}{dx}(f(g(x))) = \frac{d}{dx}(y) =$

$= \frac{dy}{dx}$ and the Chain Rule tells us that this equals $f'(g(x))\cdot g'(x) =$

$f'(u)\cdot g'(x) = \frac{d}{du}(f(u))\cdot\frac{d}{dx}(g(x)) = \frac{d}{du}(y)\frac{d}{dx}(u) = \frac{dy}{du}\cdot\frac{du}{dx}$. So, another way to state the Chain Rule is:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}}$$

MNEMONIC: A (slightly inaccurate, but) useful way to remember this form of the chain rule is "cancelling du": $\frac{dy}{\cancel{du}}\cdot\frac{\cancel{du}}{dx} = \frac{dy}{dx}$. "du" isn't really a variable on its own, but this can help for recall.

EXAMPLES: Find $f'(x)$ by using a change of coordinates.

① $f(x) = (x^2+3x+1)^{100}$

Write $y = (x^2+3x+1)^{100}$ and $u = x^2+3x+1$. Then $y = u^{100}$ and

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (100u^{99})(2x+3) = 100(x^2+3x+1)^{99}(2x+3)$$

as before.

② $f(x) = \cos(x^2+2x)$.

Let $y = \cos(x^2+2x)$ and $u = x^2+2x$. Then $y = \cos(u)$ and

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (-\sin(u))(2x+2) = -(2x+2)\sin(x^2+2x)$$

again, as before.