§2.7

MAIN CONCEPTS

1. The Chain Rule, Revisited & Implicit Differentiation
2. Inverse Functions
3. Logs and Exponentials

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1. We've seen how to take the derivative of $y=\sqrt{x}$ by writing $y=x^{1/2}$ and using the power rule. But we discovered the power rule using whole-number exponents ($x^1, x^2, x^3, \dots$). Why should the same rule work for fractional powers?

The chain rule gives us a way to answer this. Recall that one way to use the chain rule was to write $\frac{dy}{dx} = \frac{du}{dx}\cdot\frac{dy}{du}$. Dropping the "y" gives $\boxed{\frac{d}{dx} = \frac{du}{dx}\cdot\frac{d}{du}}$. So, if we need to differentiate with respect to a different variable, we can! We replace $\frac{d}{dx}$ with $\frac{du}{dx}\cdot\frac{d}{du}$.

For example, if $y=\sqrt{x}$ like above, then $y^2=x$ and we can use $\frac{d}{dx}$ on both sides: $\frac{d}{dx}(y^2) = \frac{d}{dx}(x)$. The right side is straightforward: $\frac{d}{dx}(x)=1$. But for the left we can only use the power rule if the letters match [i.e. $\frac{d}{dx}(x^2)=2x$ or $\frac{d}{du}(u^2)=2u$ or $\frac{d}{dy}(y^2)=2y$, but $\frac{d}{dx}(y^2)$ is all mixed up].

So, we use the chain rule to switch $\frac{d}{dx}$ for $\frac{dy}{dx}\cdot\frac{d}{dy}$:

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x)$$

$$\frac{dy}{dx}\cdot\frac{d}{dy}(y^2) = 1$$

$$\frac{dy}{dx}(2y) = 1$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

So, we were able to solve for $\frac{dy}{dx}$ after all! Our final answer should be back in terms of x though, so we plug in $y=\sqrt{x}$ again:

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

Notice that this shows us that the power rule really does work here:

$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}\cdot\frac{1}{x^{1/2}} = \frac{1}{2}\cdot\frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

gives the same answer.

This method in general is called "implicit differentiation" because you take derivatives implicitly, rather than first solving for y explicitly. The strategy is:

1. Relate x & y (or whatever variables you have). Try to make the relationship "nice" (square roots, division, etc. are hard and can get messy).
2. Apply $\frac{d}{dx}$ to both sides. Use derivative rules to simplify.
3. For any mismatches $\left[\frac{d}{dx}(y^2), \text{ etc.}\right]$ use the chain rule.
4. Solve for $\frac{dy}{dx}$ algebraically
5. If possible, put your answer in terms of x again.

EXAMPLES: Find $\frac{dy}{dx}$.

1. $x^2 + y^3 = 2x + 1$

This already looks pretty good, so let's apply $\frac{d}{dx}$:

$$\frac{d}{dx}(x^2+y^3) = \frac{d}{dx}(2x+1) \longrightarrow 2x + \frac{d}{dx}(y^3) = 2$$

$$\rightarrow 2x + \frac{dy}{dx}\cdot\frac{d}{dy}(y^3) = 2 \longrightarrow 2x + \frac{dy}{dx}(3y^2) = 2$$

$$\rightarrow \frac{dy}{dx}(3y^2) = 2-2x \longrightarrow \boxed{\frac{dy}{dx} = \frac{2-2x}{3y^2}}$$

If we want, we can solve for y in the original: $y^3 = 2x+1-x^2$ $\quad y = (2x+1-x^2)^{1/3}$

and plug it in: $\frac{dy}{dx} = \frac{2-2x}{3(2x+1-x^2)^{2/3}}$ but this is fairly messy and not necessarily better than the boxed solution.

2. $y = \frac{x}{y+1}$

This doesn't look too bad, but we will need the quotient rule on the right. Multiplying by $(y+1)$ on both sides removes the denominator: $y(y+1) = x \longrightarrow y^2+y = x$. Now we differentiate: $\frac{d}{dx}(y^2+y) = \frac{d}{dx}(x) \Rightarrow \frac{dy}{dx}\frac{d}{dy}(y^2+y) = 1$

$$\longrightarrow \frac{dy}{dx}(2y+1) = 1 \quad \text{so} \quad \boxed{\frac{dy}{dx} = \frac{1}{2y+1}}$$

③ $\sin(xy) = 1$

This is simple enough, so apply $\frac{d}{dx}$:

$\frac{d}{dx}(\sin(xy)) = \frac{d}{dx}(1)$

$\cos(xy)\frac{d}{dx}(xy) = 0$ [Chain rule: we have an "inside function" xy]

$\cos(xy)\left[\frac{d}{dx}(x)\cdot y + x\frac{d}{dx}(y)\right] = 0$ [Product rule]

$1\cdot y + x\frac{dy}{dx} = 0$ [Divided by $\cos(xy)$ on both sides]

$x\frac{dy}{dx} = -y$

$$\boxed{\frac{dy}{dx} = -\frac{y}{x}}$$

2. As a special application, we can use implicit differentiation to find the derivatives of inverse functions. We start with inverse trig.

EXAMPLES: Find $\frac{dy}{dx}$ in terms of x only.

① $y = \arcsin(x)$.

Remember arcsin (or $\sin^{-1}$) "undoes" sin. So we can rewrite this as

$\sin(y) = x$. Now we use $\frac{d}{dx}$:

$\frac{d}{dx}(\sin(y)) = \frac{d}{dx}(x) \longrightarrow \frac{dy}{dx}\cdot\frac{d}{dy}(\sin(y)) = 1 \longrightarrow \frac{dy}{dx}\cos(y) = 1.$

So $\frac{dy}{dx} = \frac{1}{\cos(y)}$, but we want it in terms of x. Since $\cos^2(y)+\sin^2(y) = 1$,

we get $\cos^2(y) = 1-\sin^2(y) \longrightarrow \cos(y) = \sqrt{1-\sin^2(y)}$. But $\sin(y) = x$,

so $\cos(y) = \sqrt{1-(x)^2} = \sqrt{1-x^2}$. So $\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}}$

② $y = \arctan(x)$.

Same idea: $\tan(y) = x$ so

$\frac{d}{dx}(\tan(y)) = \frac{d}{dx}(x) \longrightarrow \frac{dy}{dx}\frac{d}{dy}(\tan(y)) = 1$

$\longrightarrow \frac{dy}{dx}\sec^2(y) = 1 \longrightarrow \frac{dy}{dx} = \frac{1}{\sec^2(y)}$.

Using the trig identity $\sec^2(y) = 1+\tan^2(y) = 1+(\tan(y))^2 = 1+(x)^2 = 1+x^2$

gives $\boxed{\frac{dy}{dx} = \frac{1}{1+x^2}}$

In fact, these two are special enough to memorize:

FACT: $\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$ and $\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$.

[3] Finally, we can do the same with e^x and $\ln(x)$. In a lab, you'll investigate $\ln(x)$ and learn that $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$. We can use this to solve $\frac{d}{dx}(e^x)$:

EXAMPLE: Find $\frac{d}{dx}(e^x)$.

Write $y = e^x$, so we're looking for $\frac{dy}{dx}$. But then $\ln(y) = \ln(e^x) = x$, so we can use implicit differentiation:

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(x)$$

$$\frac{dy}{dx}\frac{d}{dy}(\ln(y)) = 1$$

$$\frac{dy}{dx}\left(\frac{1}{y}\right) = 1$$

$$\frac{dy}{dx} = y$$

I.e. $\frac{dy}{dx} = e^x$ so e^x is its own derivative.

Again: let's memorize these.

FACT: $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ & $\frac{d}{dx}(e^x) = e^x$.