§2.8

Main Concepts

1. Related rates — the idea
2. Related rates — in practice

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[1] Many quantities depend on time — location, the volume of a cup as it fills up, the price of an object, and the angle of the sun, to name a few. We can ask how quickly they change, which is, of course, the derivative. But it's rare to know these exactly as a function of t [when can you really say price $= 3\cos(t) + 7$?].

The idea, then, is that some rates are measurable and we can calculate the ones that aren't measurable by relating them to ones that are. For example, if you're filling a bottle from a water cooler, the volume comes out in strange amounts, but if you keep a close eye on the height (water level), you can work out how quickly the volume is increasing from that.

[2] In practice, this looks like implicit differentiation: to find $\frac{dy}{dt}$, relate y to x, differentiate, and use knowledge about $\frac{dx}{dt}$. This suggests a process:

Related Rates: Process

(A) Name your variables. If possible having just 2 is ideal — the one you know and the one you're looking for.

(B) Translate English to math. Figure out how to write what you know in symbols.

(C) Relate your variables. Write an equation that includes your two variables [In practice, this comes from some external source (geometry, theorems, physical laws, scientific principles, etc.). Since this class doesn't assume you know any external things (biology, physics, economics, etc.), these will either be given or based in geometry (area of a rectangle/triangle, volum of a box, pythagorean theorem, etc.)]

(D) Apply $\frac{d}{dt}$ to both sides and use implicit differentiation / chain rule to simplify.

(E) Plug in your knowns and solve for the unknown rate.

EXAMPLES:

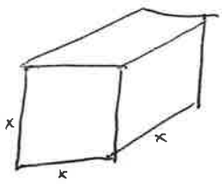
① An ice cube is melting! Initially, it has volume $27\,cm^3$, and as it melts, all 3 sides shrink at equal rates (so it remains a cube). Measuring the water dripping off, you find that initially the cube is losing $1\,cm^3$ per minute. How quickly is the side length shrinking at this moment?

Let's go through our steps:

(A) We know things about the volume and want to know about the side lengths, so let's name them: V = volume, x = side length.

(B) The given information is that initially, $\boxed{V=27}$ and $\boxed{\frac{dV}{dt} = -1}$. This second fact is a rate ($\frac{dV}{dt}$ = rate at which volume changes over time) and we've made it negative because the ice cube is losing volume.

(C) The volume of a box is length × width × height. For a cube, these are the same, so $V = (x)(x)(x) = x^3$.



(D) Now differentiate! $\frac{d}{dt}(V) = \frac{d}{dt}(x^3)$

$$\frac{dV}{dt} = \frac{dx}{dt} \cdot \frac{d}{dx}(x^3)$$

$$\frac{dV}{dt} = \frac{dx}{dt} \cdot (3x^2)$$

(E) Finally, we plug in. $(-1) = \frac{dx}{dt} \cdot (3x^2)$. But we still have x. To get x, plug into the equation from (C): $V = x^3$ so $27 = x^3$ so $\boxed{x = 3}$

So: $(-1) = \frac{dx}{dt} \cdot 3(3)^2 = 27 \cdot \frac{dx}{dt}$ and so $\boxed{\frac{dx}{dt} = -\frac{1}{27}}$. In other words, the side length is shrinking at $\frac{1}{27}$ cm per minute (or ~2.2 cm/hr).

② In the same situation as above, how quickly is the ~~side length~~ surface area shrinking?

(A) Now we have side length and surface area to work with (you could also do volume & surface area, but it might be harder). So x = side length and A = surface area.

(B) From the previous problem, we learned $x = 3$ and $\frac{dx}{dt} = -\frac{1}{27}$.

(C) Each face of a cube is a square of area length × width = $(x)(x) = x^2$. There are six sides, so $\boxed{A = 6x^2}$

(D) Differentiate! $\frac{d}{dt}(A) = \frac{d}{dt}(6x^2) \longrightarrow \frac{dA}{dt} = \frac{dx}{dt}\frac{d}{dx}(6x^2)$

$$\longrightarrow \frac{dA}{dt} = \frac{dx}{dt} \cdot (12x).$$

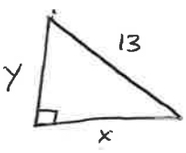
(E) Finally, plug in: $\frac{dA}{dt} = \left(-\frac{1}{27}\right) \cdot (12(3)) = -\frac{36}{27} = -\frac{4}{3}$.

So, the area is shrinking by $\boxed{4/3\ cm^2 \text{ per minute}}$.

(3) A ladder is ~~slipping down a wall~~ propped against a wall. It's 13 ft long and suddenly starts to slip down the wall and along the ground. When the top of the ladder is 5 ft above the ground, it's falling at 2 ft/sec. How quickly is the bottom of the ladder moving away from the wall?

(A) In order to name our variables, a picture can be useful. Let's draw one that's descriptive and one that's "math-y." Then we see the two variables of interest: x = base, y = height. Notice that we could name the hypotenuse (z = hypotenuse, maybe?) but it's not changing, so we don't need to.





(B) Now, we know that when $y = 5$, $\frac{dy}{dt} = -2$. Again, we make $\frac{dy}{dt}$ negative since y is decreasing.

(C) We want to relate x & y. This is a right triangle, so the simplest is the pythagorean theorem: $x^2 + y^2 = 13^2$ (i.e. $x^2 + y^2 = 169$).

(D) Differentiate: $\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(169) \longrightarrow \frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = 0 \longrightarrow$

$\longrightarrow \frac{dx}{dt}\frac{d}{dx}(x^2) + \frac{dy}{dt}\frac{d}{dy}(y^2) = 0 \longrightarrow \frac{dx}{dt}(2x) + \frac{dy}{dt}(2y) = 0.$

(E) Now we plug in, but like in problem (1), we need to know x as well as y, $\frac{dy}{dt}$.

So first, we plug into (C) itself: $x^2+y^2=169 \longrightarrow x^2+(5)^2=169 \longrightarrow x^2+25=169$

$\longrightarrow x^2 = 144 \longrightarrow x = 12$. Then we plug into (D):

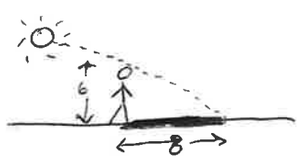
$$\frac{dx}{dt}(2x) + \frac{dy}{dt}(2y) = 0 \longrightarrow \frac{dx}{dt}(2(12)) + \left(\frac{dy}{dt}\right)(2(5)) = 0 \longrightarrow$$

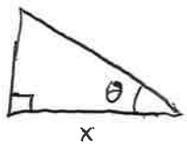
$$\longrightarrow 24\frac{dx}{dt} - 20 = 0 \longrightarrow \boxed{\frac{dx}{dt} = \frac{20}{24} = \frac{5}{6}.}$$

I.e. it's slipping ~0.83 ft/sec away from the wall.

(4) A 6-foot tall person is standing outside as the sun is setting. If their shadow is currently 8 feet long, how quickly is it growing?

(A) Let's start again with pictures. We want to know the shadow length so x = shadow length. The problem is that we aren't given any rates! But reflecting on it, we realize that the angle of the sun changes consistently. So, we'll call θ the angle between the sun and horizon.





(B) Then, we know that currently $x = 8$ and $\frac{d\theta}{dt} = \frac{2\pi \text{ radians}}{24 \text{ hours}}$ (full cycle every day), so

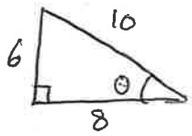
$\frac{d\theta}{dt} = -\pi/12$ (negative because the sun is setting).

(C) Then, from the right triangle, we see the relationship $\tan\theta = \frac{6}{x}$.

(D) Before differentiating, I'll rewrite the relationship to simplify: $\tan\theta = 6x^{-1}$. So:

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}(6x^{-1}) \longrightarrow \frac{d\theta}{dt}\cdot\frac{d}{d\theta}(\tan\theta) = \frac{dx}{dt}\cdot\frac{d}{dx}(6x^{-1}) \longrightarrow \frac{d\theta}{dt}\sec^2(\theta) = \frac{dx}{dt}(-6x^{-2})$$

(E) Finally, we plug in. We need θ to plug it in, but actually we only need $\sec^2(\theta)$, so we look at our triangle again. By Pythagoras:



$(\text{hypotenuse})^2 = 6^2+8^2 = 36+64 = 100$, so

hypotenuse $= 10$. But then, $\sec^2(\theta) = \frac{1}{\cos^2(\theta)} = \frac{1}{\left(\frac{\text{adjacent}}{\text{hypotenuse}}\right)^2} = \frac{1}{\left(\frac{8}{10}\right)^2} = \frac{100}{64}$.

So, putting it all together: $\frac{d\theta}{dt}\sec^2(\theta) = \frac{dx}{dt}(-6x^{-2})$

$$\rightarrow \left(-\frac{\pi}{12}\right)\left(\frac{100}{64}\right) = \frac{dx}{dt}\left(-6(8)^{-2}\right)$$

$$\rightarrow -\frac{100\pi}{12\cdot 64} = \frac{dx}{dt}\left(-\frac{6}{64}\right)$$

$$\frac{100\pi}{12} = 6\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{100\pi}{12\cdot 6} = \frac{25\pi}{18} \approx 4.36 \text{ ft/hr}$$

or about 0.87 inches every minute.