§2.9

## Main Concepts

1. Linear approximation
2. Error propagation.

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1. In this section, we look at two practical uses of the derivative. The first is to approximate a (potentially complicated) function by a simple one. Let's say we're trying to understand the function $f(x)$ for $x \approx a$. In other words, $x = a+h$ for $h$ "small." Then

$$f(x) = f(a+h) = f(a+h) - f(a) + f(a) = \frac{f(a+h)-f(a)}{h} \cdot h + f(a) \approx f'(a) \cdot h + f(a)$$

where we used the definition of the derivative for the "$\approx$." Putting it back in terms of $x$:

$$f(x) \approx f'(a) \cdot h + f(a) = f'(a) \cdot (x-a) + f(a).$$

This gives a good approximation as long as $h$ is small, i.e. as long as $x$ is near $a$. This is nice because the approximation is linear (of the form $mx+b$), and we understand linear functions the best.

**Best Linear Approximation:** For $x$ near $a$, $f(x) \approx f(a) + f'(a)(x-a)$

**Example:** Find the best linear approximation to $f(x) = \sin(x)$ near $x = \pi$. Use this to estimate $\sin(3)$.

In other words, we have $a = \pi$. Then $f(a) = f(\pi) = \sin(\pi) = 0$. Also $f'(x) = \cos(x)$ and $f'(a) = \cos(\pi) = -1$. So near $\pi$, $\sin(x) \approx (0) + (-1)(x-\pi) = \pi - x$.

Thus, $\sin(3) \approx \pi - 3 = 0.14159\ldots$

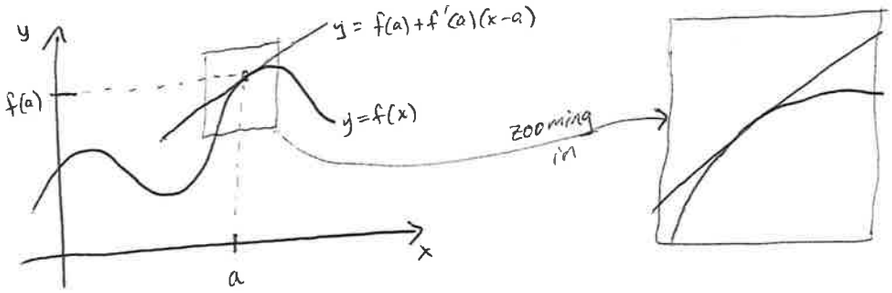
**Example:** Use a linear approximation to estimate $\sqrt{10}$.

Let $f(x) = \sqrt{x}$. We're looking for some value "$a$" so that $a \approx 10$ and $f(a)$, $f'(a)$ are easy to compute. First, note $f(x) = x^{1/2}$, so $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$, so for $f(a)$, $f'(a)$ to be "simple," we want $\sqrt{a}$ to be "nice." In other words, we're looking for a perfect square near 10. 9 is pretty close and $9 = 3^2$, so let's use $a = 9$:

$f(a) = \sqrt{9} = 3$ $\quad f'(a) = \frac{1}{2\sqrt{9}} = \frac{1}{2 \cdot 3} = \frac{1}{6}$ $\quad$ so $f(x) \approx 3 + \frac{1}{6}(x-9)$. Thus

$$\sqrt{10} = f(10) \approx 3 + \frac{1}{6}(10-9) = 3 + \frac{1}{6} = \frac{19}{6} = 3.166\ldots$$

You may notice that we've seen this expression $(f(a) + f'(a)(x-a))$ before. Namely, the graph of $y = f(a) + f'(a)(x-a)$ has slope $f'(a)$ and passes through the point $(a, f(a))$. In other words, $y = f(a) + f'(a)(x-a)$ is the equation of the tangent line to $y = f(x)$ at the point $(a, f(a))$.




[We can see here the comparison: being tangent makes the function and line roughly equal, at least for $x \approx a$].

2 Linear approximation is useful when you know the input exactly, but the output is easier to understand somewhere nearby. Other times, instead, we only know the input approximately. For example, perhaps you measure the side length of a cube as 4 cm, but your ruler has an error of $\pm 0.1$ cm. Then the volume is $4^3 = 64\ cm^3$, but what's the error?

We can work this out with linear approximations. Let's say $x = a + \Delta x$, where $a$ is the measurement and $\Delta x$ is the error. If $f(x)$ is the desired output, then $f(a)$ is the expected output, but $\Delta y = f(x) - f(a)$ is the error. With linear approximation:

$$\Delta y = f(x) - f(a) \approx [f(a) + f'(a)(x-a)] - f(a) = f'(a)[(a+\Delta x) - a] = f'(a)\Delta x.$$

Recall that we can write derivatives as $\frac{dy}{dx}$. Dividing by $\Delta x$ and using this rewrite gives:

$$\frac{\Delta y}{\Delta x} \approx f'(a) = \frac{dy}{dx}$$ This is easy to remember:

**ERROR**: If $y$ depends on $x$ and $x$ is measured up to an error of $\Delta x$, then $y$ has error $\Delta y$ satisfying $\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$.

**EXAMPLES:**

① Find the error in the volume above.

Here, $V = x^3$, $x = 4$, and $\Delta x = 0.1$. $\frac{dV}{dx} = 3x^2$, so $\frac{\Delta V}{\Delta x} \approx 3x^2$, i.e. $\Delta V \approx 3x^2 \Delta x$.

Plugging in gives $\Delta V \approx 3(4)^2(0.1) = 3 \cdot 16 \cdot 0.1 = 4.8$.

So the volume is $64 \pm 4.8\ cm^3$

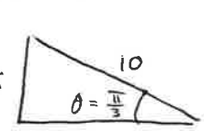
② What is the surface area, with error, of the same cube?

Now $A = 6x^2$ since there are 6 sides that are squares of side length $x$. So $\frac{dA}{dx} = 12x$ and

$\Delta A \approx \frac{dA}{dx} \cdot \Delta x = 12(4)(0.1) = 4.8$ and $A = 6(4)^2 = 96$ so the surface area is

$96 \pm 4.8\ cm^2$.

③ A right triangle has hypotenuse of length 10in and one angle is measured to be $\frac{\pi}{3}$ radians, with an error of $\pm 0.01$ radians. What is the length & error in the opposite side to that angle?




We know $\sin(\theta) = \frac{x}{10}$ so $x = 10\sin(\theta)$. Thus,

$\frac{\Delta x}{\Delta \theta} \approx \frac{dx}{d\theta} = 10\cos(\theta)$. Plugging in $\theta = \frac{\pi}{3}$ and $\Delta\theta = 0.01$ gives

$x = 10\sin(\frac{\pi}{3}) = 10 \cdot \frac{\sqrt{3}}{2} = 5\sqrt{3}$ and $\Delta x = 10\cos(\frac{\pi}{3}) \cdot (0.01) = 10(\frac{1}{2})(0.01) = 0.05$.

So the side is $5\sqrt{3} \pm 6.71$ inches, $\pm$ 0.05 inches.