§3.1

Main Concepts

1. Extreme value theorem
2. Critical points

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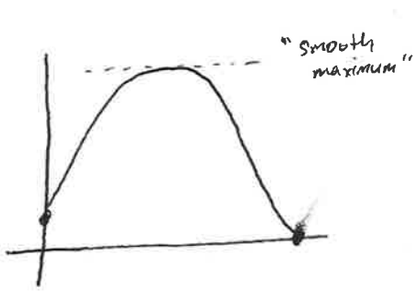
1. If $f(x)$ is a function we care about, we've seen how $f'(x)$ measures the rate at which $f(x)$ changes. Another category of question we can ask is more <u>global</u>: what can we say about $f(x)$ as x varies over a whole range of inputs. One of the first questions in this direction is on <u>extrema</u> — maxima and minima. In other words, when is $f(x)$ as big or small as possible? What is this max/min value? What value(s) of x makes this extremum occur?

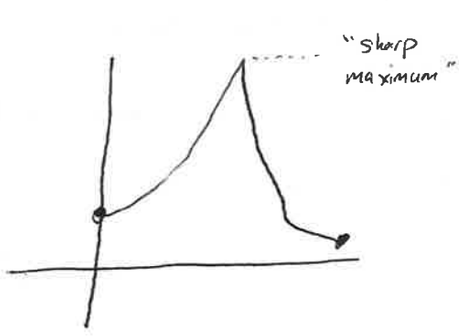
To begin, we ask what kind of conditions we need to even guarantee that extrema exist. E.g. the function $f(x) = 2x+3$ has no maximum (just plug in bigger x) or minimum (plug in smaller x). But if we restrict our domain to a <u>bounded interval</u>, i.e. $a \le x \le b$ for some $a \le b$, then we will get a max/min.

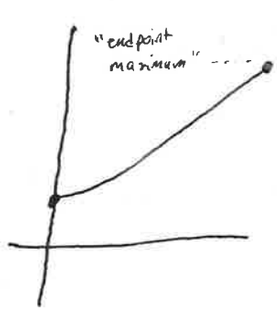
But bounded isn't enough: the function $f(x) = \frac{1}{x}$ has a minimum if $x \in (0,7]$ (i.e. $0 < x \le 7$), but no maximum. Even though x is bounded, $\frac{1}{x} \to \infty$ as $x \to 0$ so it can't have a maximum. It turns out that the issue is the "open" end at 0. If we have a closed interval $[a,b]$, then this kind of thing can't happen, since $\lim_{x \to a^+} f(x) = f(a)$ is defined (so not $\pm\infty$). In summary, we have:

Extreme Value Theorem: If $f(x)$ is a continuous function on a <u>closed</u> and <u>bounded</u> domain, then $f(x)$ has a maximum & minimum value.

2. This is nice, but it's an example of what we call in mathematics an <u>existence theorem</u>. It tells you that something exists (a max/min) but not how to find it. For that, we need more analysis. Thinking through some examples, there are really 3 kinds of "maxima" that can appear on a graph.







In fact, these are the only possibilities. Let's give them names.

DEFINITION: Say $f(x)$ is a function and x is some point in the domain $[a,b]$ (so $a \le x \le b$).

(I) If $f'(x)=0$, we say x is a stationary point of $f(x)$.

(II) If $f'(x)$ does not exist, we say x is a singular point of $f(x)$.

(III) If $x=a$ or $x=b$, we say x is an endpoint of $f(x)$.

If any of the above are true, we say x is a critical point of $f(x)$

EXAMPLES: For each function $f(x)$, find all critical points of $f(x)$.

① $f(x) = x^2 - 2x$ on $[-3, 7]$.

First, we look for stationary points: $f'(x) = 2x-2$ so a stationary point is when

$2x-2=0 \longrightarrow 2x=2 \longrightarrow \boxed{x=1}$

Second, we look for singular points, but $f'(x)=2x-2$ is always defined.

Finally, we have the endpoints $\boxed{x=-3}$ and $\boxed{x=7}$ These are our (three) critical points.

② $f(x) = x + \frac{2}{x-4}$ on $[0, 10]$.

Here $f(x) = x + 2(x-4)^{-1}$ so by the Chain rule:

$f'(x) = 1 - 2(x-4)^{-2}\cdot(1) = 1 - \frac{2}{(x-4)^2}$. So, we get

$1 - \frac{2}{(x-4)^2} = 0 \longrightarrow 1 = \frac{2}{(x-4)^2} \longrightarrow (x-4)^2 = 2 \longrightarrow x-4 = \pm\sqrt{2}$

$\longrightarrow \boxed{x = 4 \pm \sqrt{2}}$, are two critical points.

Note $f'(x)$ does not exist when $\boxed{x=4}$ since we'd then divide by zero.

Finally we have the endpoints $\boxed{x=0}$ and $\boxed{x=10}$. These are the five critical points.

③ $f(x) = x^3$ on $[1, 3]$

We have $f'(x) = 3x^2$. $f'(x)=0 \longrightarrow 3x^2=0 \longrightarrow x=0$, but this is not in the domain $(1 \le x \le 3)$ so it's not a critical point. $f'(x)$ always exists, so there's no singular points. So, in this case we only have the endpoints $\boxed{x=1}$ and $\boxed{x=3}$.

The key result is what I stated above: these are the only possibilities

CRITICAL POINT THEOREM: If $f(x)$ is maximized or minimized at some x, then x is a critical point of $f(x)$.

This gives a strategy: if the domain is closed & bounded we know the max/min exist; to find it, list all the critical points and test each one.

EXAMPLES: Find the minimum and maximum of $f(x)$ on the specified domain.

① $f(x) = x^2 - 2x$ on $[-3, 7]$.

This was example ① above. We already found the critical points, so let's test them:

$f(1) = (1)^2 - 2(1) = 1 - 2 = \boxed{-1}$ $\quad f(-3) = (-3)^2 - 2(-3) = 9 + 6 = 15$

$f(7) = (7)^2 - 2(7) = 49 - 14 = \boxed{35}$

Since -1 is the smallest, the minimum value is -1, occuring at $x = 1$. Since 35 is the biggest, the maximum value is 35, occuring at $x = 7$.

② $f(x) = x + \frac{2}{x-4}$ on $[0, 10]$

This was ②. Testing each gives:

$f(4+\sqrt{2}) = (4+\sqrt{2}) + \frac{2}{(4+\sqrt{2})-4} = 4 + \sqrt{2} + \frac{2}{\sqrt{2}} = 4 + \sqrt{2} + \sqrt{2} = 4 + 2\sqrt{2}$.

$f(4-\sqrt{2}) = (4-\sqrt{2}) + \frac{2}{(4-\sqrt{2})-4} = 4 - \sqrt{2} + \frac{2}{-\sqrt{2}} = 4 - \sqrt{2} - \sqrt{2} = 4 - 2\sqrt{2}$

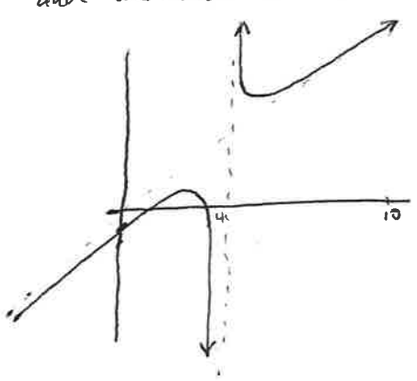
$f(4) = 4 + \frac{2}{4-4} = 4 + \frac{2}{0}$ DNE

$f(0) = (0) + \frac{2}{(0)-4} = \frac{2}{-4} = -\frac{1}{2}$

$f(10) = (10) + \frac{2}{(0)-4} = 10 + \frac{2}{6} = \frac{31}{3}$.

We might be tempted to say the max is $31/3$ and the min is $-1/2$. But notice the $x = 4$ is a problem: since it doesn't exist, our domain isn't really closed + bounded [it's actually $[0,4) \cup (4,10]$ instead of $[0,10]$ as claimed] and we know it'll have a vertical asymptote there. So, it has $\boxed{\text{no max/min}}$.

Here's a sketch of the graph to help explain this issue.



③ $f(x) = x^3$ on $[1, 3]$.

Again, this was ③. Testing the two points gives

$f(1) = (1)^3 = 1$ and $f(3) = (3)^3 = 27$. So $f(x)$ has a max of 27 ($@x=3$) and a min of 1 ($@x=1$).

④ $f(x) = \sqrt[3]{x^2}$ on $[-1, 8]$

As before, let's start by finding critical points. We can rewrite $f(x) = (x^2)^{1/3} = x^{2/3}$, so $f'(x) = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-1/3} = \frac{2}{3} \cdot \frac{1}{x^{1/3}} = \frac{2}{3\sqrt[3]{x}}$.

So $f'(x) = 0 \rightarrow \frac{2}{3\sqrt[3]{x}} = 0 \rightarrow 2 = 0(3\sqrt[3]{x}) = 0$ is never true. So there are

no stationary points. However if $\boxed{x=0}$ then we have $f'(x) = \frac{2}{3\sqrt[3]{0}} = \frac{2}{0}$ DNE so this is a singular point. Finally we have $\boxed{x=-1}$ and $\boxed{x=8}$ as endpoints.

Testing them all gives: $f(0) = \sqrt[3]{(0)^2} = \sqrt[3]{0} = 0$

$f(-1) = \sqrt[3]{(-1)^2} = \sqrt[3]{1} = 1$ and $f(8) = \sqrt[3]{(8)^2} = \sqrt[3]{64} = 4$.

So $f(x)$ has a max of 4 at $x=8$ and a min of 0 at $x=0$. Notice it was important to have the singular point here.

(5) $f(x) = x^3 - 4x^2 + 5x + 10$ on $[0, 2]$

First, $f'(x) = 3x^2 - 8x + 5 = 0$ gives $(x-1)(3x-5) = 0 \rightarrow \boxed{x=1}$ and $\boxed{x=\frac{5}{3}}$ are critical points. $f'(x)$ always exists, so no singular points. Finally, $\boxed{x=0}$ and $\boxed{x=2}$ are endpoints. So we have

$f(0) = (0)^3 - 4(0)^2 + 5(0) + 10 = 0 - 0 + 0 + 10 = 10$

$f(1) = (1)^3 - 4(1)^2 + 5(1) + 10 = 1 - 4 + 5 + 10 = 12$

$f(5/3) = (\frac{5}{3})^3 - 4(\frac{5}{3})^2 + 5(\frac{5}{3}) + 10 = \frac{125}{27} - \frac{100}{9} + \frac{25}{3} + 10 = \frac{320}{27} \approx 11.85\ldots$

$f(2) = (2)^3 - 4(2)^2 + 5(2) + 10 = 8 - 16 + 10 + 10 = 12$

So we have a minimum of 10 at $x=0$ and a maximum of 12 at <u>two points</u> $x=1$ and $x=2$.

Again, we can sketch the graph to see this.

