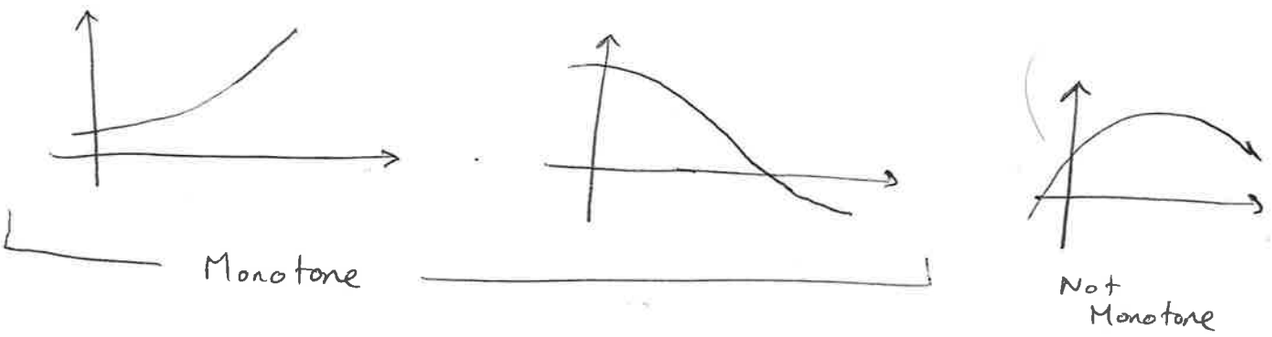
§3.2

Main Concepts

1. Monotonicity
2. Concavity

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1. We've seen that the derivative can tell us if a function is increasing or decreasing by the sign of the derivative: positive ⟷ increasing, negative ⟷ decreasing. This is the idea of monotonicity: a function is monotone if it is increasing or decreasing (and so not monotone if it does both and "wiggles").



Example: For $f(x) = x^2 - 4x + 10$, describe when $f(x)$ is increasing/decreasing.

We can calculate: $f'(x) = 2x - 4$. Then $f(x)$ is increasing when

$$f'(x) > 0 \longleftrightarrow 2x - 4 > 0 \longleftrightarrow 2x > 4 \longleftrightarrow \boxed{x > 2}$$

and similarly, it's decreasing when

$$f'(x) < 0 \longleftrightarrow 2x - 4 < 0 \longleftrightarrow 2x < 4 \longleftrightarrow \boxed{x < 2}.$$

This is nice, but inequalities can be hard to work with. Notice, though, that in the above example, the behavior switched (decreasing → increasing) at $x = 2$ and $f'(2) = 2(2) - 4 = 0$. This is, of course, not a coincidence; since $f'(x)$ is continuous, switching from negative to positive requires passing through zero. This can help us focus in on the correct places.

Example: For $f(x) = x^3 - 6x^2 + 9x - 11$, describe when $f(x)$ is increasing/decreasing.

Again we start by writing $f'(x) = 3x^2 - 12x + 9$. Deciding when this is positive or negative can be tough, but inspired by the previous paragraph, let's figure out where it is zero by factoring, and look for "behavior switching":

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3).$$

So, if things change, they'll change at $x=1$ and $x=3$. This gives three regions:

$x<1$	$1<x<3$	$x>3$
$(-\infty, 1)$	$(1, 3)$	$(3, \infty)$

We'll consider each. For $x<1$, $f'(x) = 3(x-1)(x-3) = 3\cdot[\text{negative}]\cdot[\text{negative}] = [\text{positive}]$
so it's increasing here.

For $1<x<3$, $f'(x) = 3(x-1)(x-3) = 3[\text{positive}]\cdot[\text{negative}] = [\text{negative}]$
so it's decreasing here.

and for $x>3$, $f'(x) = 3(x-1)(x-3) = 3\cdot[\text{positive}]\cdot[\text{positive}] = [\text{positive}]$
so it's increasing again.

This worked, but it's still fairly tough – especially doing the "sign work" at the end. But we can make life easier: we know, in the previous example, that $f'(x)$ doesn't switch signs until $x=1$ or $x=3$. So, to figure out the sign of $f'(x)$ for $x<1$, we just need to try any test point with $x<1$, and similarly for $1<x<3$ and $x>3$. We summarize these conclusions with a sign graph:

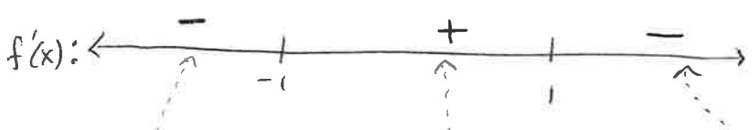


is the sign graph of the previous example.

EXAMPLE: For $f(x) = 3x^2 - x^3$, draw the sign graph for $f'(x)$ and use it to summarize when $f(x)$ is increasing/decreasing.

First, we compute $f'(x) = 3 - 3x^2 = 3(1-x^2) = 3(1-x)(1+x) = -3(x-1)(x+1)$

So $f'(x) = 0$ when $x = \pm 1$. To fill in the sign graph, we test points in between:



$f'(-2) = 3 - 3(-2)^2 = 3 - 12 = -9$

$f'(0) = 3 - 3(0)^2 = 3$

$f'(5) = 3 - 3(5)^2 = 3 - 75 = -72$

[You can choose different numbers than $-2, 0, 5$; anything $a<-1$, $-1<b<1$, and $c>1$ works].

So, $f(x)$ is decreasing for $x<-1$ and $x>1$ and

$f(x)$ is increasing for $-1<x<1$

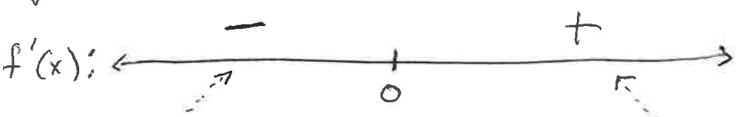
This is a nice general process — to construct a sign graph, solve for zeros (and undefined points) since these are where the sign may change, and plug in test points in between for the signs. We'll do one more example just to see the how undefined points may matter:

EXAMPLE: For $f(x) = \sqrt[3]{x^2}$, describe where $f(x)$ is increasing/decreasing.

First, rewriting $f(x) = (x^2)^{1/3} = x^{2/3}$ makes the derivative easier:

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

This has no zeros, but it isn't defined when $x=0$ (division by zero). So we have the sign graph:



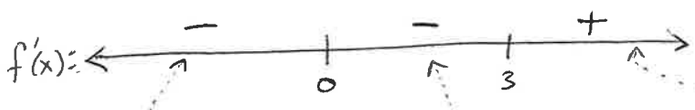
$$f'(-1) = \frac{2}{3\cdot\sqrt[3]{-1}} = \frac{2}{3(-1)} = -\frac{2}{3} \qquad f'(8) = \frac{2}{3\cdot\sqrt[3]{8}} = \frac{2}{3(2)} = \frac{1}{3}$$

So $f(x)$ increases for $x>0$ and decreases for $x<0$.

Notice that if we didn't include the discontinuity point ($x=0$) we would have missed the sign change. Finally, a note of caution — all of our sign graphs have flipped [(+ − + − ⋯) or (− + − + ⋯)] but this isn't necessary.

EXAMPLE: For $f(x) = x^4 - 4x^3 + 11$, describe where $f(x)$ is increasing/decreasing.

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) \qquad \text{so } f'(x)=0 \text{ at } x=0 \text{ and } x=3.$$



$$f'(-1) = 4(-1)^3 - 12(-1)^2 = -4 - 12 = -16$$

$$f'(1) = 4(1)^3 - 12(1)^2 = 4 - 12 = -8$$

$$f'(4) = 4(4)^3 - 12(4)^2 = 256 - 192 = 64$$

So $f(x)$ increases for $x>3$ and decreases for $x<3$.

2 Now, let's discuss concavity. This describes whether a ~~line~~ graph is "curving" or not- zero concavity is a line, positive concavity is like a cup, and negative concavity is like a hill. More typically, we write:



"Concave down"



"No concavity"



"Concave up"

How can we tell these apart? The key point is that in the first one, the slope keeps decreasing,

even going from positive to negative, and in the last one, the slope keeps increasing. Meanwhile, the middle one has constant slope. In other words:

SUMMARY: If $f''(x) > 0$, $f(x)$ is concave up

$f''(x) < 0$, $f(x)$ is concave down

But this is just like monotonicity! So we have the same strategy: solve for $f''(x) = 0$ (or undefined) and make a sign graph to identify concavity.

EXAMPLES: For each, describe where $f(x)$ is concave up/down.

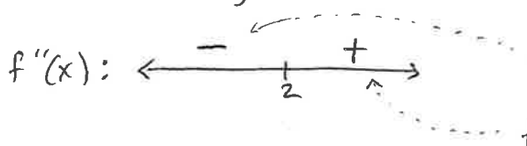
① $f(x) = x^2 - 4x + 10$.

Here, $f'(x) = 2x - 4$ and $f''(x) = 2$, which is always positive, so $f(x)$ is always concave up.

② $f(x) = x^3 - 6x^2 + 9x - 11$

Now, $f'(x) = 3x^2 - 12x + 9$ and $f''(x) = 6x - 12 = 6(x-2)$.

So $f''(x) = 0$ only when $x = 2$. We make the sign graph:



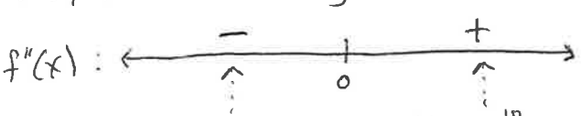
$f''(0) = 6(0) - 12 = -12$

$f''(3) = 6(3) - 12 = 18 - 12 = 6$

So $f(x)$ is concave up for $x > 2$ and concave down for $x < 2$.

③ $f(x) = \sqrt[3]{x^5}$

Rewriting $f(x) = x^{5/3}$ gives $f'(x) = \frac{5}{3}x^{2/3}$ and $f''(x) = \frac{10}{9}x^{-1/3} = \frac{10}{9 \cdot \sqrt[3]{x}}$; this has no roots, but a discontinuity at $x = 0$:

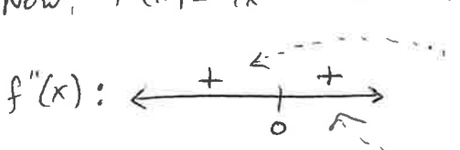


$f''(-1) = \frac{10}{9\sqrt[3]{-1}} = -\frac{10}{9}$ $\qquad$ $f''(1) = \frac{10}{9 \cdot \sqrt[3]{1}} = \frac{10}{9}$

So $f(x)$ is concave up for $x > 0$ and concave down for $x < 0$.

④ $f(x) = x^4$

Now, $f'(x) = 4x^3$ and $f''(x) = 12x^2$ which is zero when $x = 0$ only.



$f''(-1) = 12(-1)^2 = 12$

$f''(1) = 12(1)^2 = 12$

So, $f(x)$ is concave up everywhere.