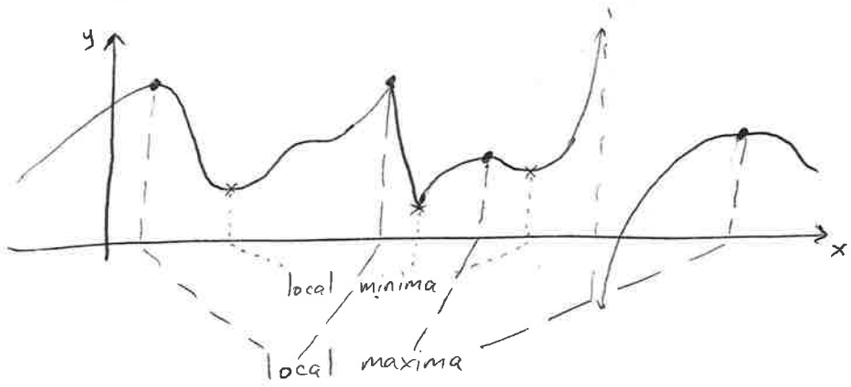
§3.3

Main Concepts

1. Local extrema
2. 1st derivative test
3. 2nd derivative test

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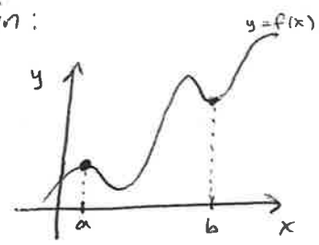
1 In §3.1, we considered extreme values, the maximum and minimum of a function on its entire domain. We now zoom in to look at local extrema, function values where it is maximized or minimized, but only for "nearby" values of x.



We notice immediately that local extrema occur at the same places as global extrema so:

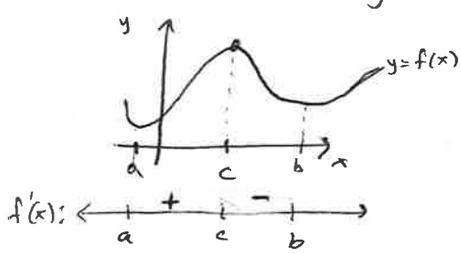
(Local) Critical Point Theorem: If $f(x)$ has a local extremum at x, then x is a critical point of $f(x)$.

In other words, to find ~~extrema~~ local extrema, we begin as before — find the critical points. However, deciding whether a critical point is a local max, a local min, or a red herring is harder than for global ones — we can't just plug in values since a local max can be even less than a local min:

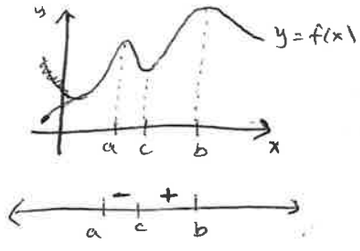


$x=a$ is a local max
$x=b$ is a local min
but $f(a) < f(b)$.

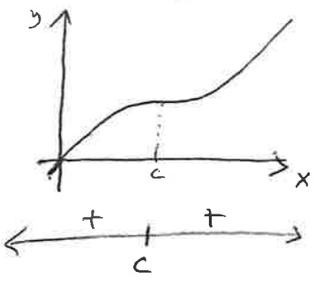
2 Our first strategy to resolve this is the 1st derivative test. If $x=c$ is a critical point, and $f(x)$ is increasing for $a<x<c$ and decreasing for $c<x<b$, then $x=c$ must be a local max. We can visualize this with the graph, as well as the sign graph for $f'(x)$:

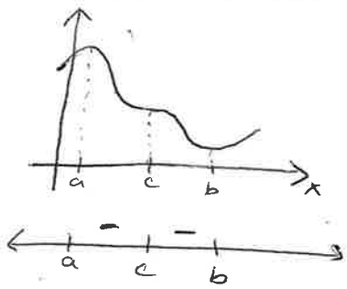


Similarly, the reverse is true for a local min:



On the other hand, the other two possibilities for the sign graph are neither local maxima nor local minima.

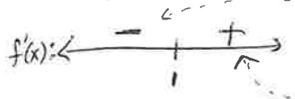




EXAMPLES: Find all critical points of $f(x)$ and classify each as a local maximum, local minimum, or as neither.

① $f(x) = x^2 - 2x + 7$

As in §3.2, we construct the sign graph of $f'(x)$: $f'(x) = 2x - 2 = 2(x-1)$ which is zero if $x = 1$.



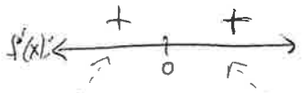
$f'(0) = 2(0) - 2 = -2$

$f'(2) = 2(2) - 2 = 2$

So, $x = 1$ is a local minimum of $f(x)$.

② $f(x) = x^3$

Ditto: $f'(x) = 3x^2 \longrightarrow x = 0$ is the critical point

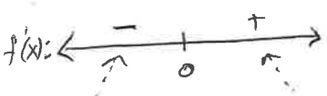


$f'(-1) = 3$ $\qquad$ $f'(1) = 3$

So $x = 0$ is <u>not</u> a local extremum.

③ $f(x) = \sqrt[3]{x^2}$

Rewrite $f(x) = x^{2/3}$, so $f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$ which has a critical point at $x = 0$ (since $f'(0)$ DNE).



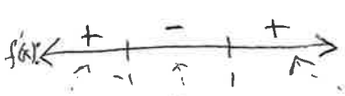
$f'(-1) = -\frac{2}{3}$ $\qquad$ $f'(1) = \frac{2}{3}$

So $x = 0$ is a local minimum of $f(x)$.

④ $f(x) = x^3 - 3x$

Now, $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1)$, so $x = 1$ and $x = -1$ are critical points.

The sign graph



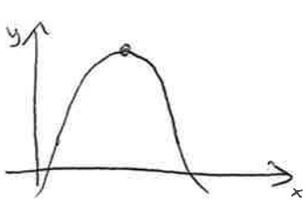
tells us that $x = -1$ is a local maximum and $x = 1$ is a local minimum.

$f'(-2) = 3(-2)^2 - 3 = 12 - 3 = 9$

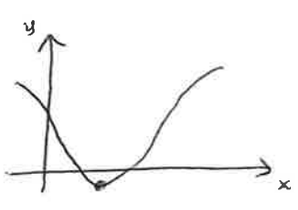
$f'(0) = 3(0)^2 - 3 = 0 - 3 = -3$

$f'(2) = 3(2)^2 - 3 = 12 - 3 = 9$

2. We have another way to identify local maxima and local minima. If $x = c$ is a ~~local~~ critical point and $f(x)$ is concave up near $x = c$, then it must be a local minimum, and similarly concave down $\longleftrightarrow$ local maximum.



Concave down
→ local maximum



Concave up
→ local minimum.

In fact, we don't even need the word "nearby" here. Instead, we have

2nd derivative test: If $f(x)$ is such that $f'(x)$ and $f''(x)$ exist, and $f'(c)=0$, then: i) If $f''(c)>0$, $f(x)$ has a local minimum at $x=c$

ii) If $f''(c)<0$, $f(x)$ has a local maximum at $x=c$.

EXAMPLES: For each critical point in the examples above, apply the 2nd derivative test and explain its conclusions, or explain why it cannot be applied.

① We had $f(x)=x^2-2x+7$, $f'(x)=2x-2$, and $x=1$ was our critical point.

Then $f''(x)=2$, so $f''(1)=2>0$ and $x=1$ is a local min for $f(x)$.

② Here $f(x)=x^3$, $f'(x)=3x^2$, and we had critical point $x=0$. Then $f''(x)=6x$ and $f''(0)=6(0)=0$. But the 2nd derivative test has no conclusion in this case.

③ We had $f(x)=x^{2/3}$ and $f'(x)=\frac{2}{3}x^{-1/3}$ with critical point $x=0$. But the 2nd derivative test only applies to stationary points ($f'(c)=0$) and not singular points ($f'(c)$ DNE).

So again, we have no conclusion.

④ Here $f(x)=x^3-3x$, $f'(x)=3x^2-3$, and we had two critical points: $x=-1$ and $x=1$.

Then $f''(x)=6x$ and $f''(-1)=6(-1)=-6$ so $x=-1$ is a local max

and $f''(1)=6(1)=6$ so $x=1$ is a local min.

These examples show that in some ways, the 2nd derivative test is less useful than the 1st derivative test — we weren't able to analyze two of the examples here. On the other hand, the advantage of the 2nd derivative test is that I don't need all the local behavior — the 1st derivative test only works if I can construct the whole sign graph.

EXAMPLE: Find any critical point of $f(x)=\frac{x^7}{7}-\frac{x^3}{3}+\frac{x^2}{2}$ and determine if it is a local maximum, minimum, or neither.

First, $f'(x)=x^6-x^2+x=x(x^5-x+1)$. To use the 1st derivative test, we need all the roots (since $f'(x)$ can change sign at any of them), but $x^5-x+1=0$ is difficult to solve. On the other hand, it is clear that $x=0$ is a critical point. Then $f''(x)=6x^5-2x+1$ and $f''(0)=6(0)^5-2(0)+1=0-0+1=1$, so $x=0$ is a local minimum of $f(x)$.