§3.4

Main Concept

II Word problems — extrema

II We often care about extrema in real-world applications: what's the maximum size of object I can afford, what's the minimum footprint I can have, or what's the least amount of energy I can expend. We now have tools to solve such problems, as follows:

(i) Name variables for the important quantities in your problem.

(ii) Write a formula for the quantity Q that you're trying to maximize/minimize in terms of your other variables.

(iii) Translate constraints in the problem to equations between your other variables.

(iv) Use these constraints + algebra to rewrite the formula in (ii) in the form $Q = f(x)$ [i.e. depending on only one variable].

(v) Identify the domain of x [e.g. look for realistic constraints — negative lengths or areas don't make sense, etc.]

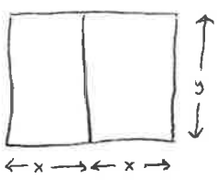
(vi) Use the tools of §3.1 and §3.3 to maximize or minimize Q.

Examples:

① I have 60 meters of fence, and want to make two identical rectangular regions that share a wall in common. What is the maximum possible area I can bound?

Let's follow our process.

(i)



Here, I draw my two identical regions, with width x and height y each, sharing a wall. I'll use A = total area.

(ii) Then $A = (2x)(y) = 2xy$.

(iii) Since I only have 60 meters of fence, I must have $4x + 3y = 60$

[there are 4 segments of length x and 3 of length y].

(iv) Now I can solve for y and eliminate it: $3y = 60 - 4x \longrightarrow y = 20 - \frac{4}{3}x$

and so $A = 2xy = 2x\left(20 - \frac{4}{3}x\right) = 40x - \frac{8}{3}x^2$.

(v) We can't have negative lengths, so $x \geq 0$ and $y \geq 0$. Plugging in for y gives

$20 - \frac{4}{3}x \geq 0 \longrightarrow 20 \geq \frac{4}{3}x \longrightarrow x \leq \frac{3}{4} \cdot 20 = 15$. I.e. we have

$0 \leq x \leq 15$ is our domain.

(vi) Since we're trying to maximize A on a closed + bounded domain, we use §3.1 ideas:

$\frac{dA}{dx} = 40 - \frac{16}{3}x = 0 \longrightarrow 40 = \frac{16}{3}x \longrightarrow x = \frac{3}{16} \cdot 40 = \frac{15}{2}$ is a critical point.

So are the endpoints $x=0$ and $x=15$. Testing them all gives:

$x=0 \longrightarrow A = 2(0)\left(20 - \frac{4}{3}(0)\right) = 0$

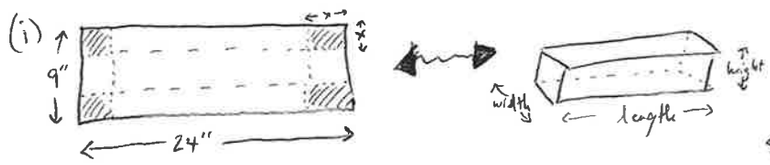
$x = \frac{15}{2} \longrightarrow A = 2\left(\frac{15}{2}\right)\left(20 - \frac{4}{3}\left(\frac{15}{2}\right)\right) = 15 \cdot 10 = 150 \leftarrow$

$x = 15 \longrightarrow A = 2(15)\left(20 - \frac{4}{3}(15)\right) = 30 \cdot 0 = 0$

So, the maximum area we can enclose is $150\ m^2$, when $x = 7.5\ m$ (and $y = 10\ m$).

(2) I have a $24'' \times 9''$ piece of cardboard. I'd like to cut squares out of the four corners and fold up the flaps to form a box with no lid. How big should I make my cuts to maximize the volume of the box?

(i)



We have to cut squares so that the flaps fold up to equal height. We'll call the side length of that square x & the volume $= V$.

(ii) Now $V = (\text{length}) \times (\text{width}) \times (\text{height})$

$= (24-2x)(9-2x)(x) = (216 - 66x + 4x^2)x = 216x - 66x^2 + 4x^3$.

(iii) There's only one other variable,

(iv) so nothing to do here!

(v) Again, negative length doesn't work, so $x \geq 0$. We also can't cut too much or our planned cuts will overlap. Indeed, $x \leq \frac{9}{2}$ because at $x = \frac{9}{2}$, they just touch. So our domain is $0 \leq x \leq \frac{9}{2}$.

(vi) Now we maximize. We have endpoints $x=0$ and $x = 9/2$. Then

$$\frac{dV}{dx} = 216 - 132x + 12x^2 = 12(x^2 - 11x + 18) = 12(x-2)(x-9).$$

So $x=2$ and $x=9$ are also critical points. But $x=9$ is not in the domain, so we ignore it. Testing all the critical points gives:

$x=0 \longrightarrow V = 216(0) - 66(0)^2 + 4(0)^3 = 0$

$x=2 \longrightarrow V = (24-2(2))(9-2(2))(2) = (20)(5)(2) = 200 \leftarrow$

$x = 9/2 \longrightarrow V = \left(24 - 2\left(\frac{9}{2}\right)\right)\left(9 - 2\left(\frac{9}{2}\right)\right)\left(\frac{9}{2}\right) = (15)(0)\left(\frac{9}{2}\right) = 0$

So, we get a max. volume of $200\ in^3$ by cutting $2'' \times 2''$ squares from each corner.