§3.6

MAIN CONCEPT

1 The Mean Value Theorem

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1 The Mean Value Theorem is a mathematical expression of the following idea: if you travel 45 miles over the course of 1 hour, then at some instant, you were traveling exactly 45 mi/hr. This seems relatively clear to us intuitively. In fact, if we think of our speed as continuous then we know that never going exactly 45 mi/hr means we were always going faster or always going slower. But then there's no way we traveled exactly 45 miles in this time.

Formally, the statement is as follows:

MEAN VALUE THEOREM: If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) then there is some c with $a<c<b$ and $f'(c) = \frac{f(b)-f(a)}{b-a}$.

First, let's see how this formalizes the above idea. Let $f(x)$ be your location after x hours. Let's say that initially, you start at 0, so $f(0)=0$. Then we said that you traveled 45 miles in 1 hour, so $f(1)=45$. Applying MVT to $[0,1]$ then gives

$$f'(c) = \frac{f(1)-f(0)}{1-0} = \frac{45-0}{1-0} = \frac{45}{1} = 45$$

for some c. But if $f(x)$ is position, then $f'(x)$ is velocity, so this says that your velocity was 45 mi/hr for some time c, which is what we said.

EXAMPLE: Convince yourself that MVT works for $f(x)=x^2-4x+10$ on $[0,6]$.

The left-hand-side of MVT is $f'(c)$. But $f'(x)=2x-4$ so $f'(c)=2c-4$.

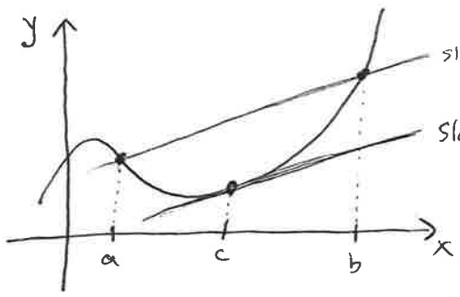
The right-hand-side is $\frac{f(b)-f(a)}{b-a} = \frac{f(6)-f(0)}{6-0} = \frac{1}{6}\left[(6^2-4\cdot 6+10)-(0^2-4\cdot 0+10)\right] =$

$$= \frac{1}{6}\left[(36-24+10)-(10)\right] = \frac{1}{6}(12) = 2.$$

So, these are equal when $2c-4=2 \longrightarrow 2c=6 \longrightarrow c=3$.

Indeed, $0<c<6$ in this case, so MVT worked.

Let's dig a bit more to understand MVT. First, let's understand it geometrically. The quantity $\frac{f(b)-f(a)}{b-a}$ should look somewhat familiar — it's $\frac{y_2-y_1}{x_2-x_1}$ which we know gives us the slope between (x_1,y_1) and (x_2,y_2). The expression $f'(c)$ is also a slope, so we should be able to "see" the two equal slopes from two parallel lines. Indeed...



...this diagram shows the two lines. We can thus rephrase MVT in words: "the average slope equals the instantaneous slope somewhere."

A second way to understand MVT is to see when it doesn't apply:

(NON-) EXAMPLES: Explain what's wrong in each application of MVT.

① "For $f(x)=|x|$, $f(-1)=|-1|=1$ and $f(2)=|2|=2$. So, by MVT, there is some c with $-1<c<2$ and $f'(c)=\frac{f(2)-f(-1)}{2-(-1)}=\frac{2-1}{2+1}=\frac{1}{3}$."

The problem here is that $f(x)$ is not differentiable on $(-1,2)$. In particular, $f'(0)$ DNE.

In fact, the conclusion is false since we know $f'(x)=\begin{cases}1 & \text{if } x>0\\ -1 & \text{if } x<0\end{cases}$ so $f'(c)\neq\frac{1}{3}$ for any c.

② "For $f(x)=\frac{1}{x^2}$, $f(-1)=\frac{1}{(-1)^2}=\frac{1}{1}=1$ and $f(2)=\frac{1}{2^2}=\frac{1}{4}$ so by MVT, there is some c with $-1<c<2$ and $f'(c)=\frac{(1/4)-(1)}{2-(-1)}=\frac{-3/4}{3}=-\frac{1}{4}$."

The issue is almost the same — $f(x)$ is not defined at $x=0$, so not continuous or differentiable on $(-1,2)$. This time, we have $f'(x)=-\frac{2}{x^3}=-\frac{1}{4}$ has a solution: $x^3=8\to x=2$ but note that $c=2$ doesn't satisfy $-1<c<2$ [it's not $\leq$].

Finally, note the following application of MVT: If $f'(x)=g'(x)$, does $f(x)=g(x)$? Well, not necessarily, since we know derivatives "kill constants." E.g. $f(x)=x^2$ and $g(x)=x^2+100$ both have $f'(x)=g'(x)=2x$, but $f(x)\neq g(x)$. However, this is the only issue — if we choose $h(x)=f(x)-g(x)$, then $h'(x)=f'(x)-g'(x)=0$. But then, if $h(a)\neq h(b)$, MVT would give:

$$h'(c)=\frac{h(b)-h(a)}{b-a}=\frac{\text{nonzero}}{\text{nonzero}}=\text{nonzero}$$ but we know $h'(c)=0$ for any c.

So, we must have $h(a)=h(b)$ for any a and b — i.e. $f(x)=g(x)+C$ for some constant.

EXAMPLE: If $f'(x)=4x$, describe all possibilities for $f(x)$.

We know $\frac{d}{dx}(2x^2)=4x$, so $f'(x)=(2x^2)'$. By the previous paragraph,

$$f(x)=2x^2+C \text{ for some constant } C.$$

We'll return to this idea in §3.8.