§3.7

MAIN CONCEPTS

1. Numerical Methods
2. Bisection Algorithm
3. Newton's Method

---

1. A common goal in mathematics is to solve problems of the form $f(x)=0$ for some function $f(x)$. We have various strategies — factoring, taking roots, formulas like the quadratic formula, etc. But sometimes, nothing you know seems to work and instead we look for numerical methods — processes that don't necessarily find the answer, but can at least approximate it. Often, this is enough — do you really need the 100th digit of π to compute the surface area of a beach ball? Here, we discuss two numerical methods and analyze their efficacy in solving problems.

2. The first method is called the Bisection Method or Bisection Algorithm (or sometimes, just bisection). The idea is as follows: if $f(x)$ is continuous, and $f(a)$ is negative while $f(b)$ is positive, then IVT guarantees that $f(x)=0$ somewhere in between. Naively, we bisect (i.e. cut in half) and guess that maybe $f(c)=0$ at the midpoint $c=\frac{a+b}{2}$. If it is, we got lucky and we're done! Otherwise, if $f(c)$ is positive, then we can try again, knowing there's a solution between a and c. Finally, if $f(c)$ is negative, then we can try again, knowing there's a solution between b and c.

EXAMPLES:

(1) Estimate a solution to $x^3-3x-5=0$, knowing that there's a solution between $x=1$ and $x=5$.

Here $f(x)=x^3-3x-5$, $f(1)=(1)^3-3(1)-5=1-3-5=-7$ and
$f(5)=(5)^3-3(5)-5=125-15-5=105$.

So there is a solution in $[1,5]$ as claimed. Let's try the midpoint $x=\frac{1+5}{2}=\frac{6}{2}=3$.

Then $f(3)=(3)^3-3(3)-5=27-9-5=13$ which is positive.

So there is a solution in $[1,3]$. Let's again try the midpoint $x=\frac{1+3}{2}=\frac{4}{2}=2$.

Then $f(2)=(2)^3-3(2)-5=8-6-5=-3$ which is negative.

So there is a solution in $[2,3]$. Try the midpoint $x=\frac{2+3}{2}=\frac{5}{2}=2.5$

Then $f(2.5)=(2.5)^3-3(2.5)-5=3.125$ which is positive.

So there's a solution in $[2,2.5]$,

We can keep going to get $[2,2.5]\to[2.25,2.5]\to[2.25,2.375]\to[2.25,2.3125]\to$
$\to[2.25,2.28125]\to\cdots$

We see that after a few steps, we can estimate the solution as $x = 2.2\cdots$. If we want more precision (more decimals), we'll just do more steps.

② Estimate $\sqrt{10}$ using bisection.

This time, no function is given. But $\sqrt{10}$ is a solution to $f(x) = x^2 - 10 = 0$, so we'll use this one. We know $\sqrt{9} = 3$, so $\sqrt{10} \approx 3$, so let's start with the guess $[3, 4]$.

Indeed $f(3) = 3^2 - 10 = 9 - 10 = -1$ is negative and $f(4) = 4^2 - 10 = 16 - 10 = 6$ is positive so there is a solution here. Try the midpoint $x = \frac{3+4}{2} = \frac{7}{2} = 3.5$ to get

$f(3.5) = (3.5)^2 - 10 = 2.25$ positive $\rightsquigarrow [3, 3.5]$, midpoint $= 3.25$

$f(3.25) = 0.5625$ positive $\rightsquigarrow [3, 3.25]$, midpoint $= 3.125$

$f(3.125) = -0.234375$ negative $\rightsquigarrow [3.125, 3.25]$

$\cdots$ [Again, more steps $\longrightarrow$ more precision].

The major advantage of bisection is that it is guaranteed to work. On the other hand, it's quite slow — even on a computer, it takes many steps to get a few more digits.

2 The next method trades these off — it's not guaranteed to work, but when it does, it gets close to the answer quite fast. The strategy is called Newton's Method, named after the mathematician/physicist Isaac Newton.

The idea is as follows: $f(x) = 0$ is hard if $f(x)$ is complicated, but if $f(x) = mx + b$ is a line, then it's easy $\leftarrow$ $mx + b = 0 \rightarrow mx = -b \rightarrow x = -\frac{b}{m}$. So, if we have a more complicated function $f(x)$, replace it with its linear approximation (see §2.9) and solve that instead. The approximation was probably not perfect, so now repeat for more accuracy.

EXAMPLES: Do the previous with Newton's method, starting with the right endpoints above.

① So here, $f(x) = x^3 - 3x - 5$ and our initial guess is $x = 5$. Remember that the linear approximation at $x = a$ is $y - f(a) = f'(a)(x - a)$ (point-slope form). Also, we're looking for $y = 0$, so we get $-f(a) = f'(a)(x - a)$. Let's use this; so note that $f'(x) = 3x^2 - 3$.

$a = 5$ $\rightsquigarrow$ $-f(5) = f'(5)(x - 5) \rightarrow -((5)^3 - 3(5) - 5) = (3(5)^2 - 3)(x - 5)$

$\rightarrow -105 = 72(x - 5) \rightarrow x - 5 = -\frac{105}{72} \approx -1.4583 \rightarrow x \approx 3.54166\cdots$

Now we repeat with this as a:

$a = 3.54166\cdots$ $\quad -f(3.54166\cdots) = f'(3.54166\cdots)(x - 3.54166\cdots) \rightarrow -28.7996 \approx 34.6302(x - 3.54166\cdots)$

$\rightarrow x \approx 3.54166\cdots - \frac{28.7996\cdots}{34.6302\cdots} \approx 2.71004.$

And we repeat:

$a = 2.71004\ldots$ $\quad -f(2.71004) = f'(2.71004)(x - 2.71004) \longrightarrow \ldots$

$\longrightarrow \ldots \longrightarrow x \approx 2.35417$

And continue: $2.35417 \longrightarrow 2.28191 \longrightarrow 2.27902 \longrightarrow 2.27902 \longrightarrow \ldots$

Notice: these last two answers appear to be the same. This suggests that in a few steps, we probably have all of those digits correct ($2.27902\ldots$). On the other hand, with bisection we only got $2.2\ldots$ with confidence.

(2) Same idea with $f(x) = x^2 - 10$ and our initial guess of $x = 4$. I'll use "x" instead of "a" now.
$f'(x) = 2x$

$x = 4$ $\quad -f(4) = f'(4)(x-4) \longrightarrow -(4^2 - 10) = (2\cdot 4)(x-4) \longrightarrow -6 = 8(x-4)$

$\longrightarrow x - 4 = -\frac{3}{4} \longrightarrow x = 3.25$

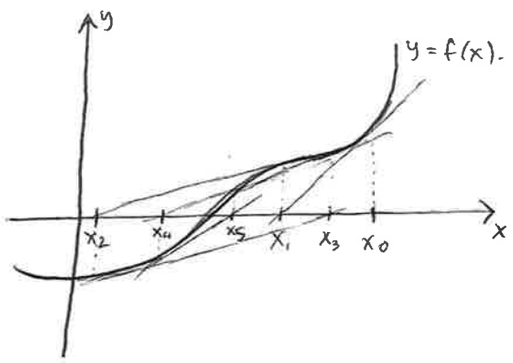
$x = 3.25$ $\quad -f(3.25) = f'(3.25)(x - 3.25) \longrightarrow -0.5625 = 6.5(x - 3.25)$

$\longrightarrow x - 3.25 = -\frac{0.5625}{6.5} \longrightarrow x = 3.1634615\ldots$

And continue: $3.1634615 \longrightarrow 3.16228 \longrightarrow 3.16228 \longrightarrow \ldots$

Here, things were even faster. After just 3 steps, we likely have the first 5 digits after the decimal: $\sqrt{10} \doteq 3.16228\ldots$

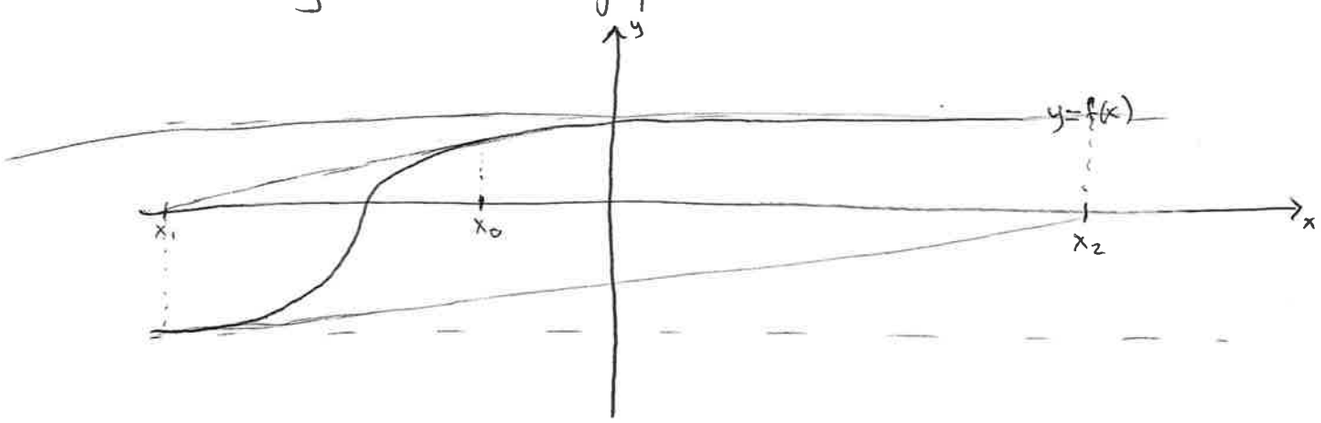
As we saw, Newton's Method gives more accuracy more quickly. However, it won't always work. To see why, let's understand it geometrically:



We'll start with an initial guess x_0. Then we draw the tangent line at x_0 to get our next guess x_1. And we repeat to get $x_2 \longrightarrow x_3 \longrightarrow x_4 \longrightarrow x_5 \longrightarrow \ldots$

It does look like the estimates are getting better.

But let's try a different graph:



After just two steps, we see that the tangent lines are just too shallow, so the estimates are getting worse and worse instead of better. In this kind of situation, you can sometimes fix things by choosing a different (better) initial guess.